

Contact Dynamics and Plastic Granular Flows

Farhang Radjaï

University of Montpellier - CNRS, LMGC, France

Stéphane Roux

LMT-Cachan, CNRS, ENS de Cachan, France

Vincent Richefeu (3SR, Grenoble), Emilien Azéma (LMGC, Montpellier),
Jean-Yves Delenne (INRA, Montpellier), Nicolas Berger (LMGC, Montpellier)



Introduction

Does a collection of **infinitely rigid** particles represent a physically valid picture of granular materials?

Plastic behavior with **no elastic domain**

DEM simulation with **no force law**

Elastic deflections can be made as small as possible: $\frac{p}{E} \rightarrow 0$

But the limit of infinitely rigid particles can not be reached in this framework:

$$\frac{p}{E} = 0 \quad !?$$

Contact Dynamics (CD)

Elastic force laws are required when a granular material is described as a **multibody** system with contact interactions.

But a granular material may also be viewed as a **multicontact** system governed by rigid-body dynamics.

Particle	Contact
Equations of motion	Kinematic constraints
Force laws	Contact laws
Particle positions and velocities	Contact forces and velocities
MD	CD

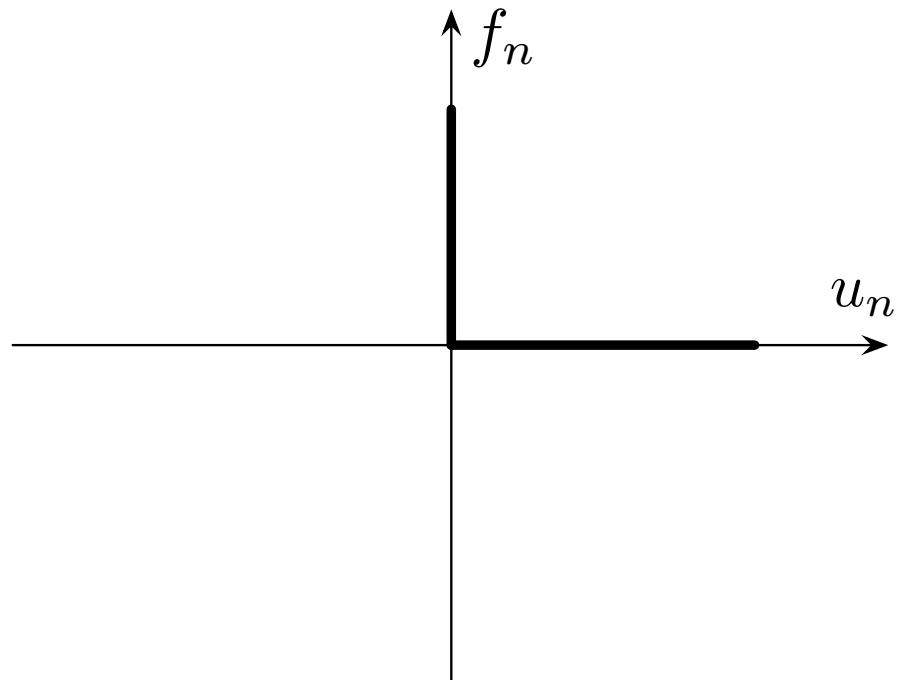
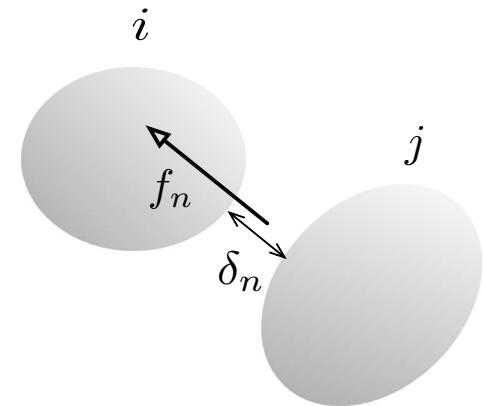
Kinematic constraints

I) Unilateral contact

$$\begin{cases} \delta_n > 0 \Rightarrow f_n = 0 \\ \delta_n = 0 \wedge \begin{cases} u_n > 0 \Rightarrow f_n = 0 \\ u_n = 0 \Rightarrow f_n \geq 0 \end{cases} \end{cases}$$

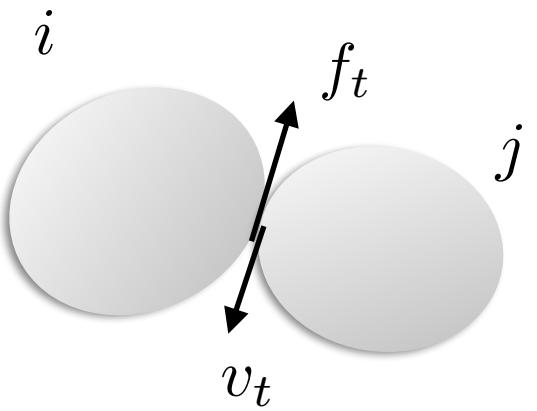
$$0 \leq f_n \perp u_n \geq 0$$

Complementarity relation



II) Coulomb friction

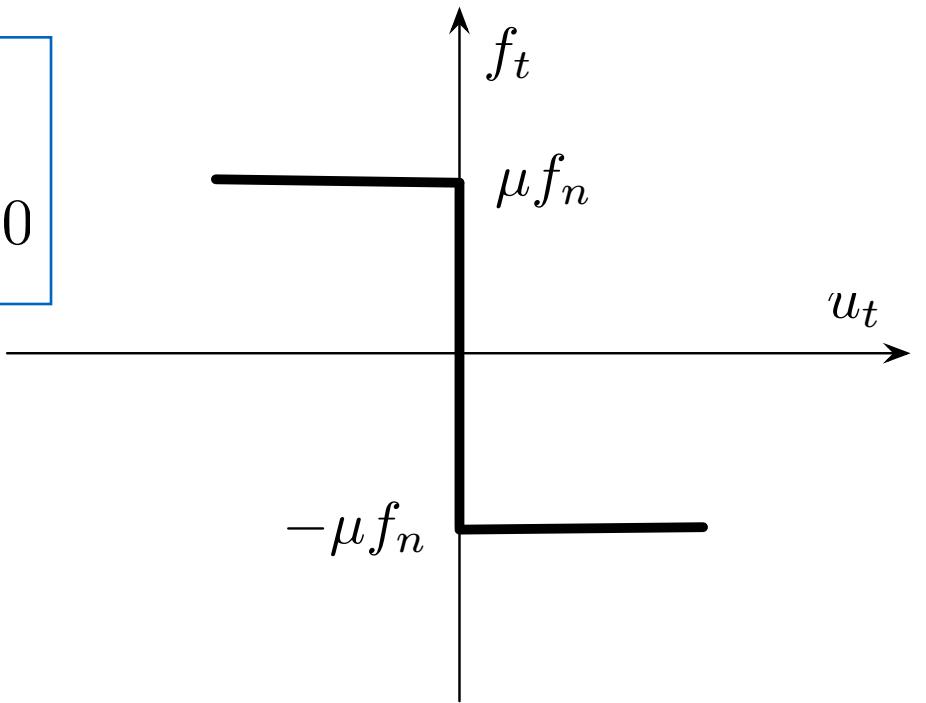
$$\begin{cases} u_t > 0 \Rightarrow f_t = -\mu f_n \\ u_t = 0 \Rightarrow -\mu f_n \leq f_t \leq \mu f_n \\ u_t < 0 \Rightarrow f_t = \mu f_n \end{cases}$$



$$0 \leq \mu f_n + f_t \perp u_t + |u_t| \geq 0$$

$$0 \leq \mu f_n - f_t \perp -u_t + |u_t| \geq 0$$

Complementarity relation



Contact dynamics equations

Particle dynamics $M(U^+ - U^-) = \delta t(F + F_{ext})$

Express dynamics in contact variables: $u_n \quad u_t \quad f_n \quad f_t$

Since the contact velocities u are linear in particle velocities, the transformation of the velocities is an affine application.

$$u = G \ U \quad G \in 2N_c \times 3N_p$$

The transformation of the contact forces to force resultants is an affine application:

$$F = H \ f \quad H = G^T \quad 3N_p \times 2N_c$$

$$\Rightarrow u^+ - u^- = \delta t H^T M^{-1} H f + \delta t H^T M^{-1} F_{ext}$$

Contact Dynamics Equations

The velocity involved in contact laws is a mean velocity weighted between the left-limit and right-limit velocities (observable velocities):

$$u_n = \frac{u_n^+ + e_n u_n^-}{1 + e_n} \quad u_t = \frac{u_t^+ + e_t u_t^-}{1 + e_t}$$

$$\begin{aligned} \frac{1 + e_n}{\delta t} (u_n^\alpha - u_n^{\alpha-}) &= \mathcal{W}_{nn}^{\alpha\alpha} f_n^\alpha + \mathcal{W}_{nt}^{\alpha\alpha} f_t^\alpha \\ &\quad + \sum_{\beta(\neq\alpha)} \{\mathcal{W}_{nn}^{\alpha\beta} f_n^\beta + \mathcal{W}_{nt}^{\alpha\beta} f_t^\beta\} + \sum_{i,j} H_n^{T,\alpha i} M^{-1,ij} F_{ext}^j \\ \Rightarrow \quad \frac{1 + e_t}{\delta t} (u_t^\alpha - u_t^{\alpha-}) &= \mathcal{W}_{tn}^{\alpha\alpha} f_n^\alpha + \mathcal{W}_{tt}^{\alpha\alpha} f_t^\alpha \\ &\quad + \sum_{\beta(\neq\alpha)} \{\mathcal{W}_{tn}^{\alpha\beta} f_n^\beta + \mathcal{W}_{tt}^{\alpha\beta} f_t^\beta\} + \sum_{i,j} H_t^{T,\alpha i} M^{-1,ij} F_{ext}^j \end{aligned}$$

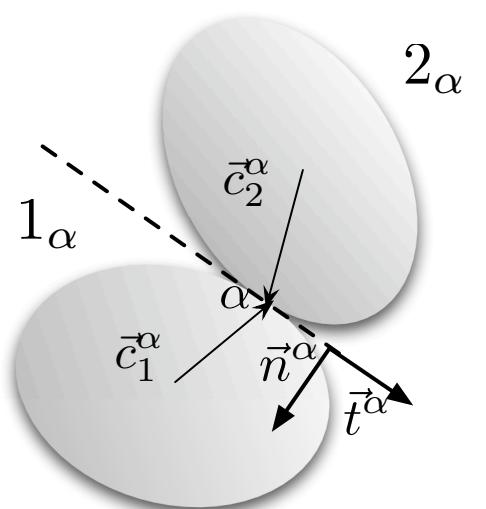
$$\mathcal{W}_{k_1 k_2}^{\alpha\beta} = \sum_{i,j} H_{k_1}^{T,\alpha i} M^{-1,ij} H_{k_2}^{j\beta} \quad \text{inverse reduced inertia}$$

$$\begin{aligned}\mathcal{W}_{nn}^{\alpha\alpha} &= \frac{1}{m_{1_\alpha}} + \frac{1}{m_{2_\alpha}} + \frac{(c_{1t}^\alpha)^2}{I_{1_\alpha}} + \frac{(c_{2t}^\alpha)^2}{I_{2_\alpha}} \\ \mathcal{W}_{tt}^{\alpha\alpha} &= \frac{1}{m_{1_\alpha}} + \frac{1}{m_{2_\alpha}} + \frac{(c_{1n}^\alpha)^2}{I_{1_\alpha}} + \frac{(c_{2n}^\alpha)^2}{I_{2_\alpha}} \\ \mathcal{W}_{nt}^{\alpha\alpha} &= \mathcal{W}_{tn}^{\alpha\alpha} = \frac{c_{1n}^\alpha c_{1t}^\alpha}{I_{1_\alpha}} + \frac{c_{2n}^\alpha c_{2t}^\alpha}{I_{2_\alpha}}\end{aligned}$$

with

$$c_{in}^\alpha = \vec{c}_i^\alpha \cdot \vec{n}^\alpha$$

$$c_{it}^\alpha = \vec{c}_i^\alpha \cdot \vec{t}^\alpha$$



Alternative representation :

$$\mathcal{W}_{nn}^{\alpha\alpha} f_n^\alpha + \mathcal{W}_{nt}^{\alpha\alpha} f_t^\alpha = (1 + e_n) \frac{1}{\delta t} u_n^\alpha + a_n^\alpha$$

$$\mathcal{W}_{tt}^{\alpha\alpha} f_t^\alpha + \mathcal{W}_{tn}^{\alpha\alpha} f_n^\alpha = (1 + e_t) \frac{1}{\delta t} u_t^\alpha + a_t^\alpha$$

with

$$a_n^\alpha = b_n^\alpha - (1 + e_n) \frac{1}{\delta t} u_n^{\alpha-} + \left(\frac{\vec{F}_{ext}^{2_\alpha}}{m_{2_\alpha}} - \frac{\vec{F}_{ext}^{1_\alpha}}{m_{1_\alpha}} \right) \cdot \vec{n}^\alpha$$

$$a_t^\alpha = b_t^\alpha - (1 + e_t) \frac{1}{\delta t} u_t^{\alpha-} + \left(\frac{\vec{F}_{ext}^{2_\alpha}}{m_{2_\alpha}} - \frac{\vec{F}_{ext}^{1_\alpha}}{m_{1_\alpha}} \right) \cdot \vec{t}^\alpha$$

offsets

and

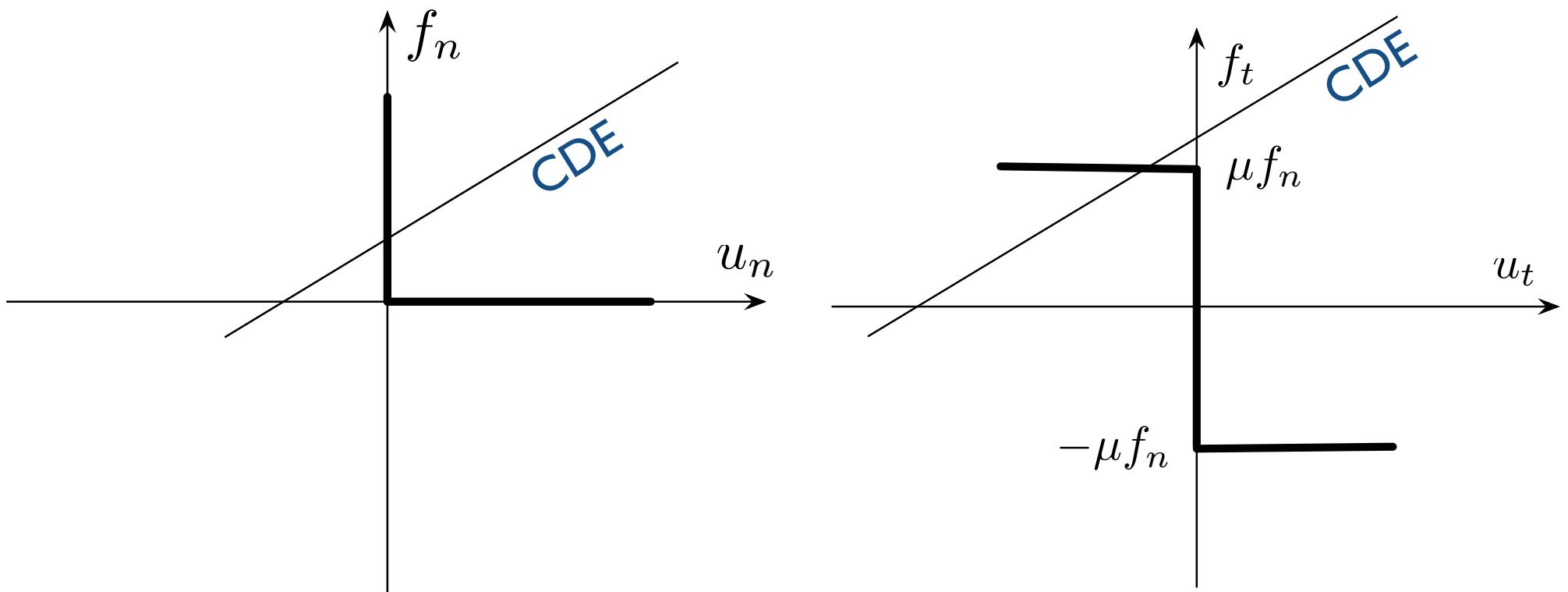
$$b_n^\alpha = \frac{1}{m_{2_\alpha}} \sum_{\beta(\neq\alpha)} \vec{f}_{2_\alpha}^\beta \cdot \vec{n}^\alpha - \frac{1}{m_{1_\alpha}} \sum_{\beta(\neq\alpha)} \vec{f}_{1_\alpha}^\beta \cdot \vec{n}^\alpha$$

coupling terms

$$b_t^\alpha = \frac{1}{m_{2_\alpha}} \sum_{\beta(\neq\alpha)} \vec{f}_{2_\alpha}^\beta \cdot \vec{t}^\alpha - \frac{1}{m_{1_\alpha}} \sum_{\beta(\neq\alpha)} \vec{f}_{1_\alpha}^\beta \cdot \vec{t}^\alpha$$

Iterative resolution

In order to solve the system of contact dynamics equations with the corresponding complementarity relations, we proceed by an iterative method which converges to the solution simultaneously for all contact forces and velocities.



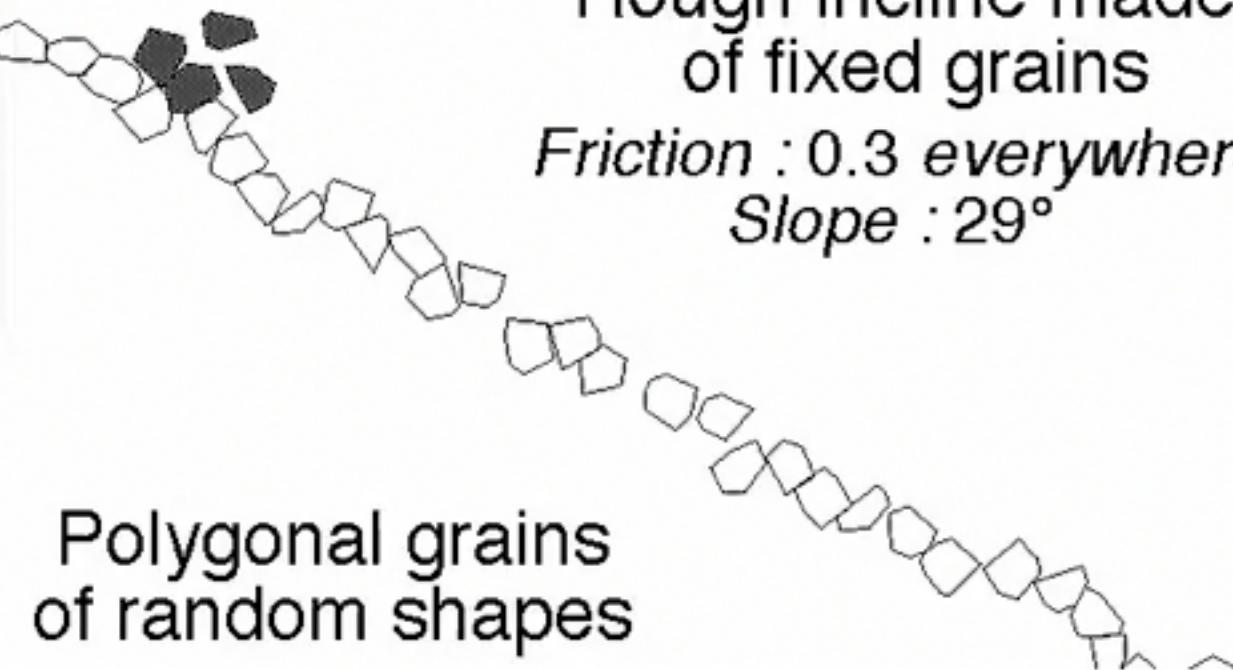
Nonsmooth Mechanics

The Contact Dynamics method was formulated by **Jean Jacques Moreau** in 1988 in the framework of **Nonsmooth Analysis** and in terms of **subdifferentials** introduced by him in 1962.



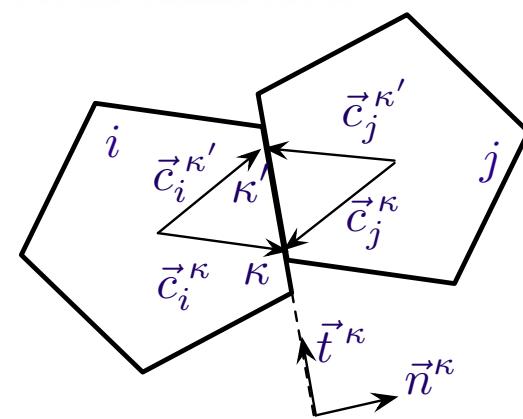
J. J. Moreau, Bounded variation in time, in Topics in Nonsmooth Mechanics edited by P.D. Panagiotopoulos and G. Strang, Birkhäuser (1988).

J. J. Moreau, Some numerical methods in multibody dynamics: application to granular materials, Eur. J. Mech.A/Solids (1994).



Polygonal grains
of random shapes

J. J. Moreau, 1996

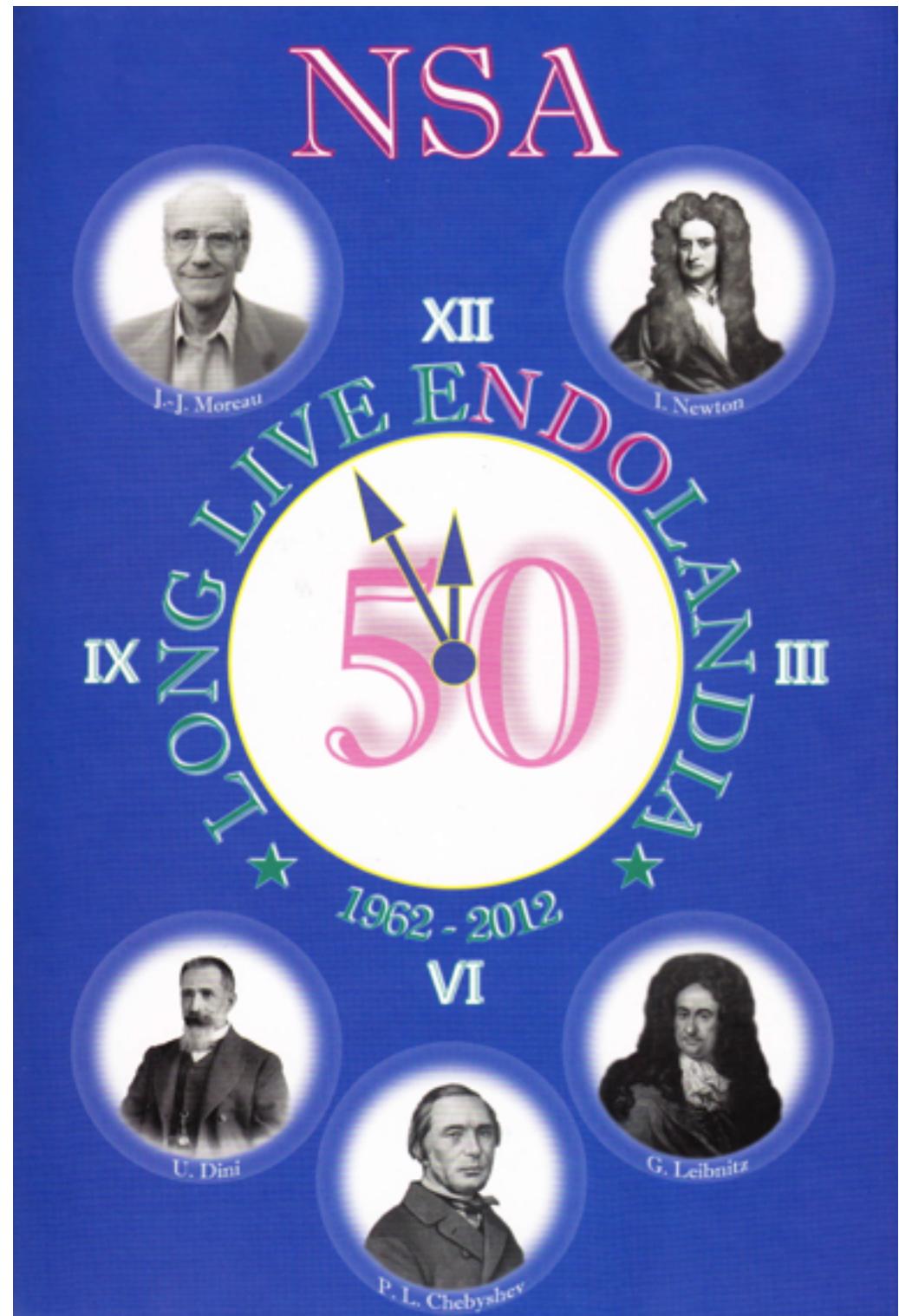


В.Ф. Дем'янов
ОПТИМИЗАЦИЯ В ЛИЦАХ.
ПУТЕШЕСТВИЕ В ЭНДОЛАНДИЮ

V.F. Demyanov
OPTIMIZATION PLACES AND FACES.
A JOURNEY TO ENDOLANDIA

NonSmooth Analysis (NSA)

- Derivative
- Gradient
- Chebyshev polynomials
- Directional derivative
- Subdifferential of convex functions (introduced in 1962 by J. J. Moreau)

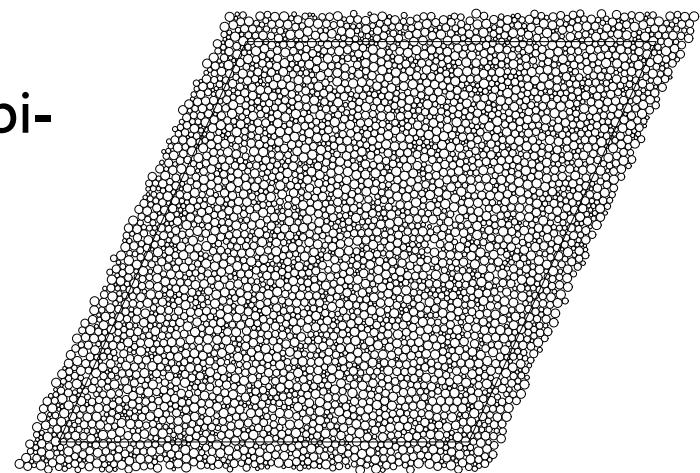
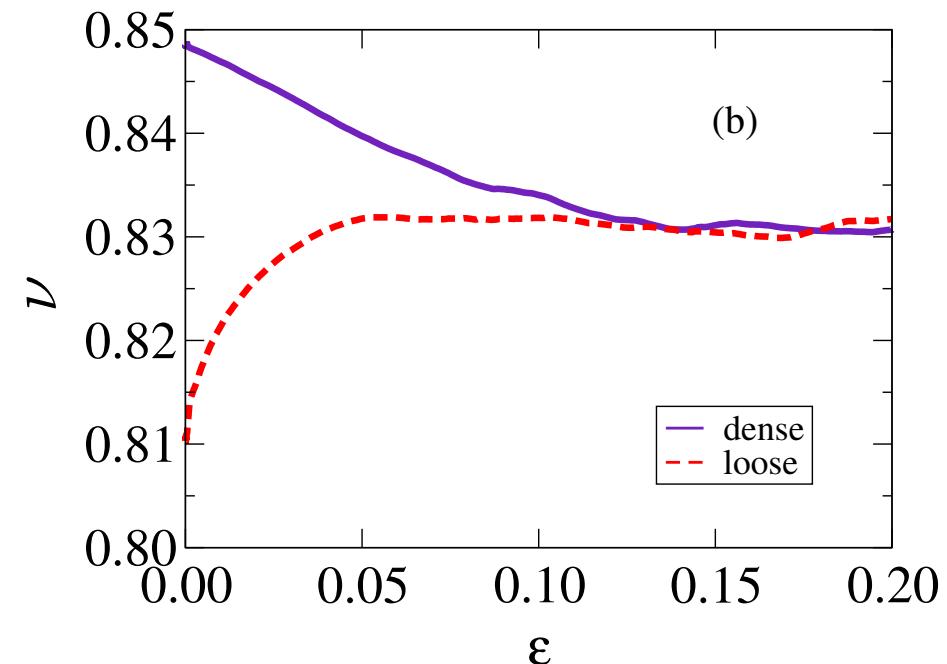
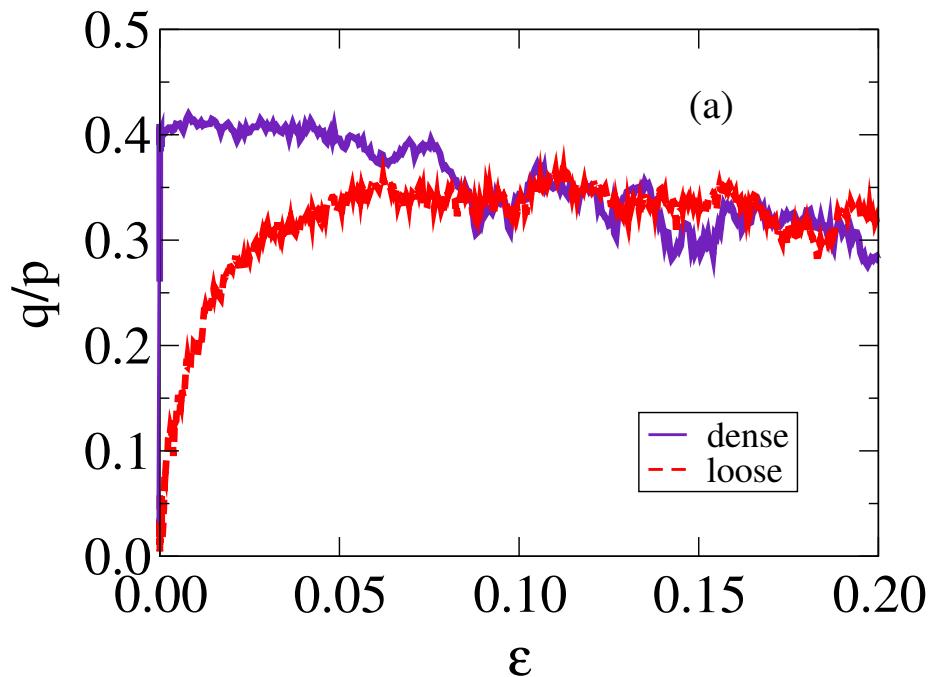


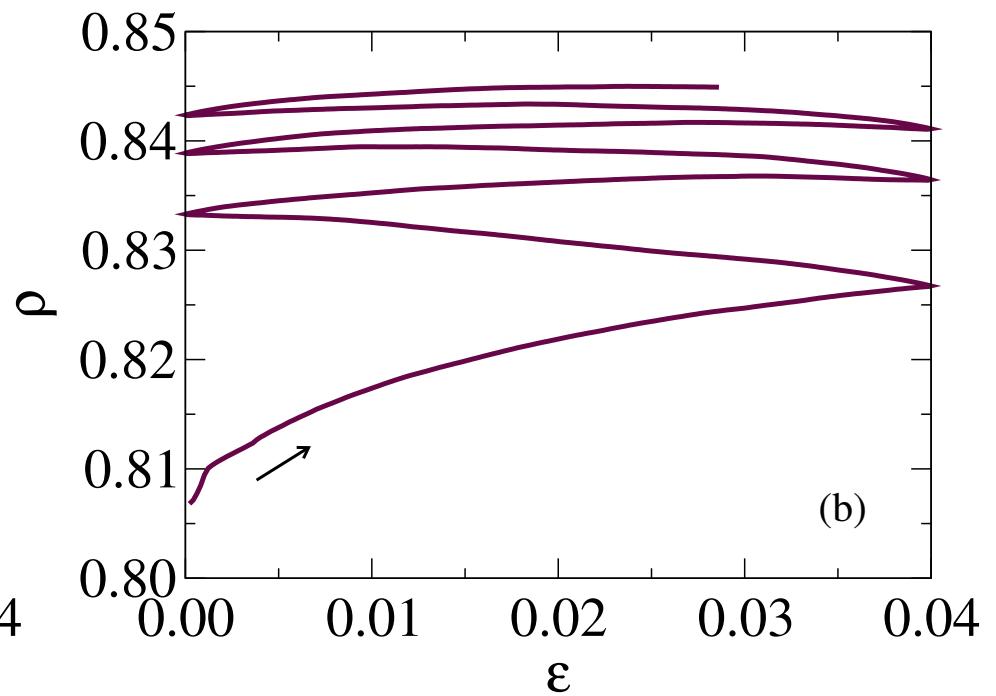
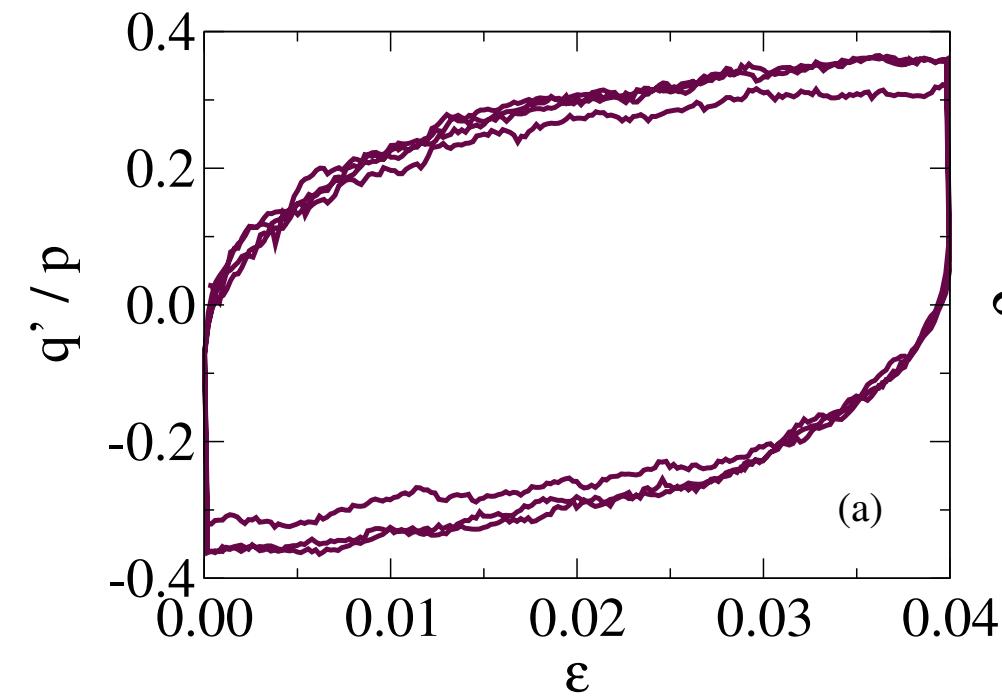
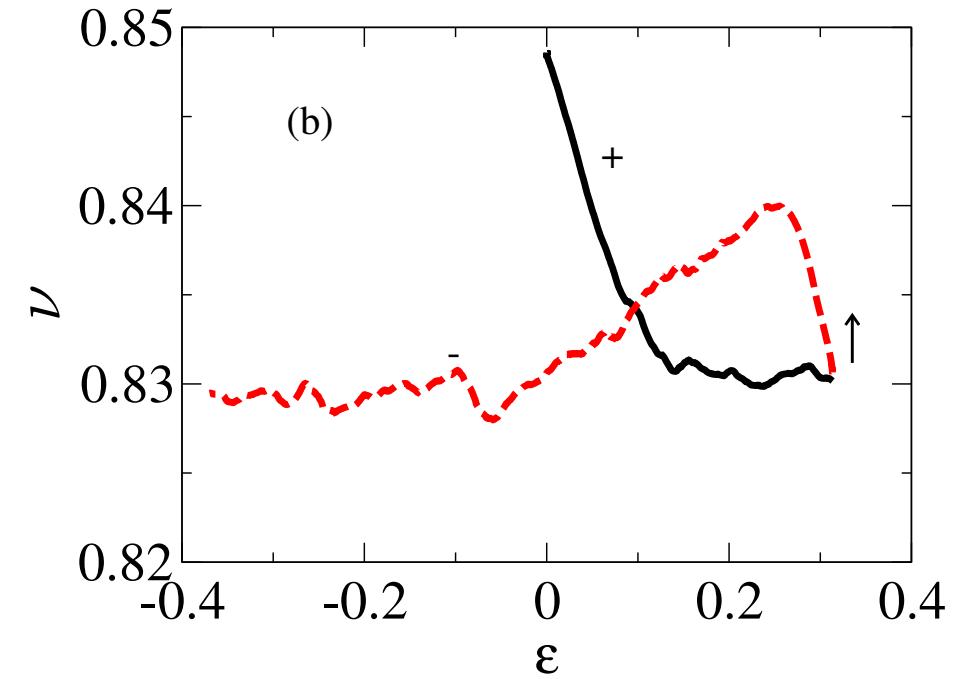
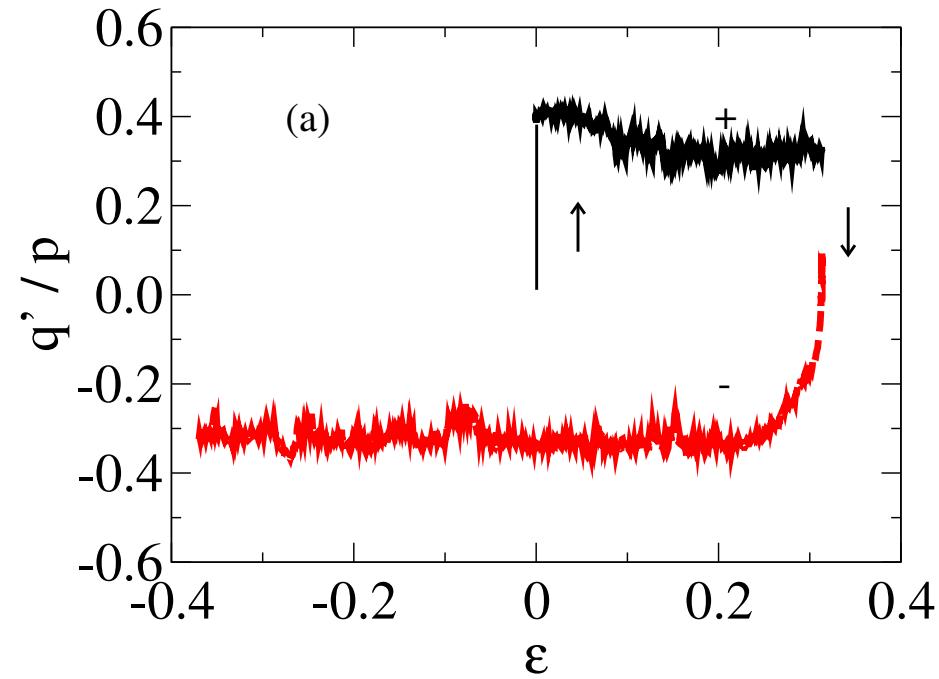
Back to rheology

CD simulations of simple shear with bi-periodic boundary conditions.

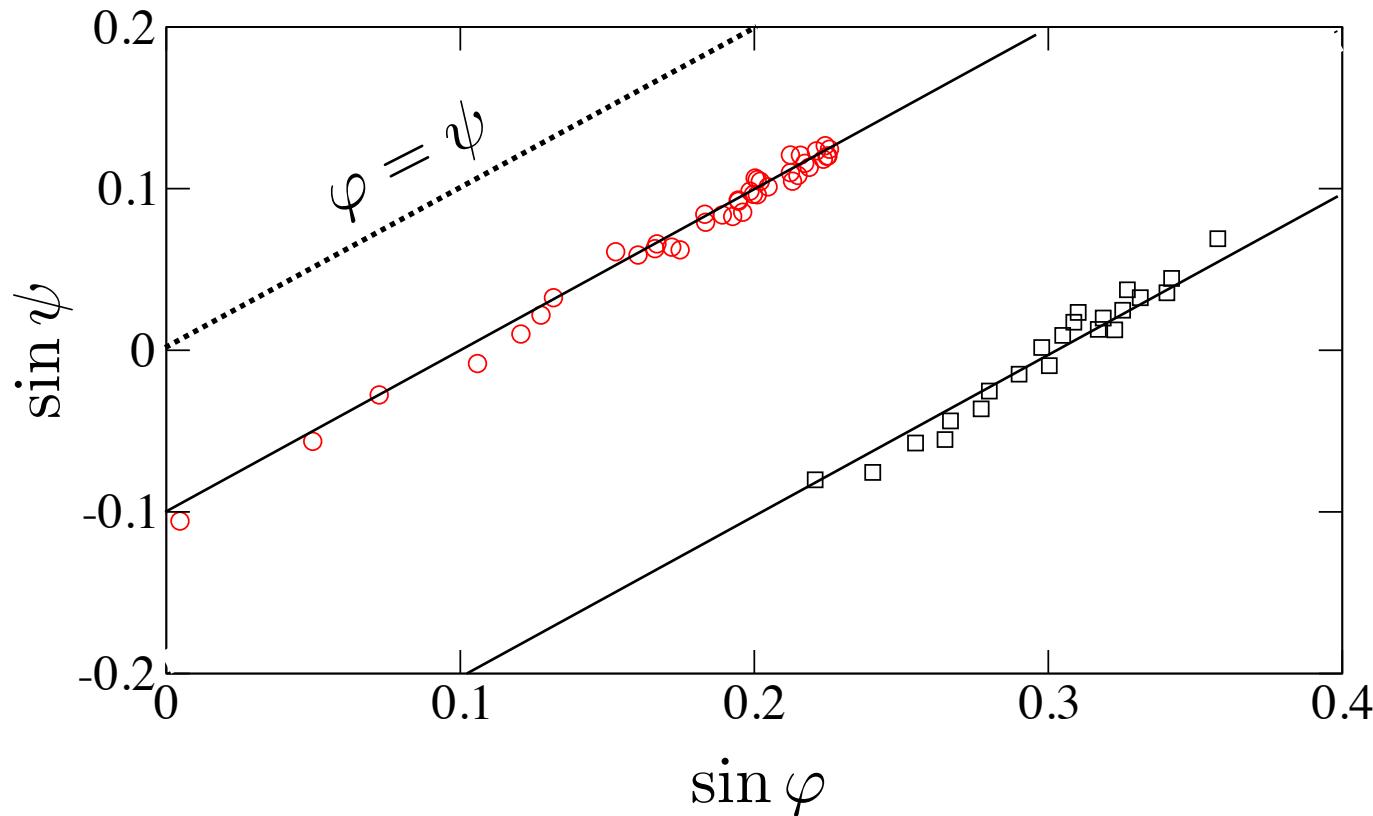
$$q = \frac{1}{2}(\sigma_1 - \sigma_2) \quad p = \frac{1}{2}(\sigma_1 + \sigma_2)$$

ν packing fraction

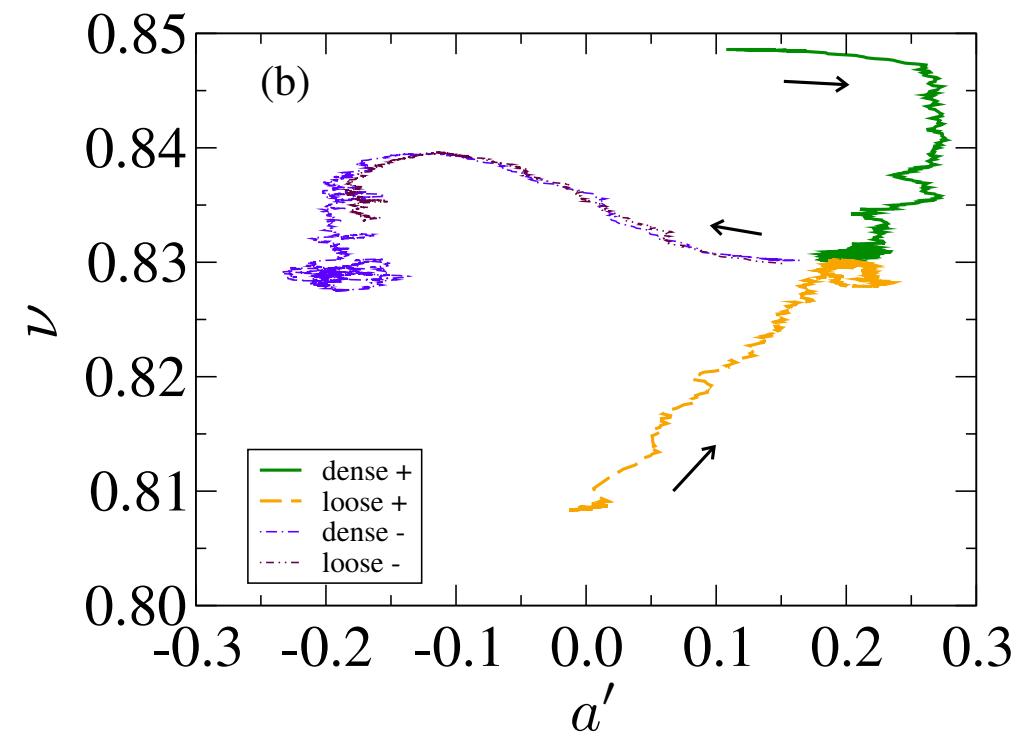
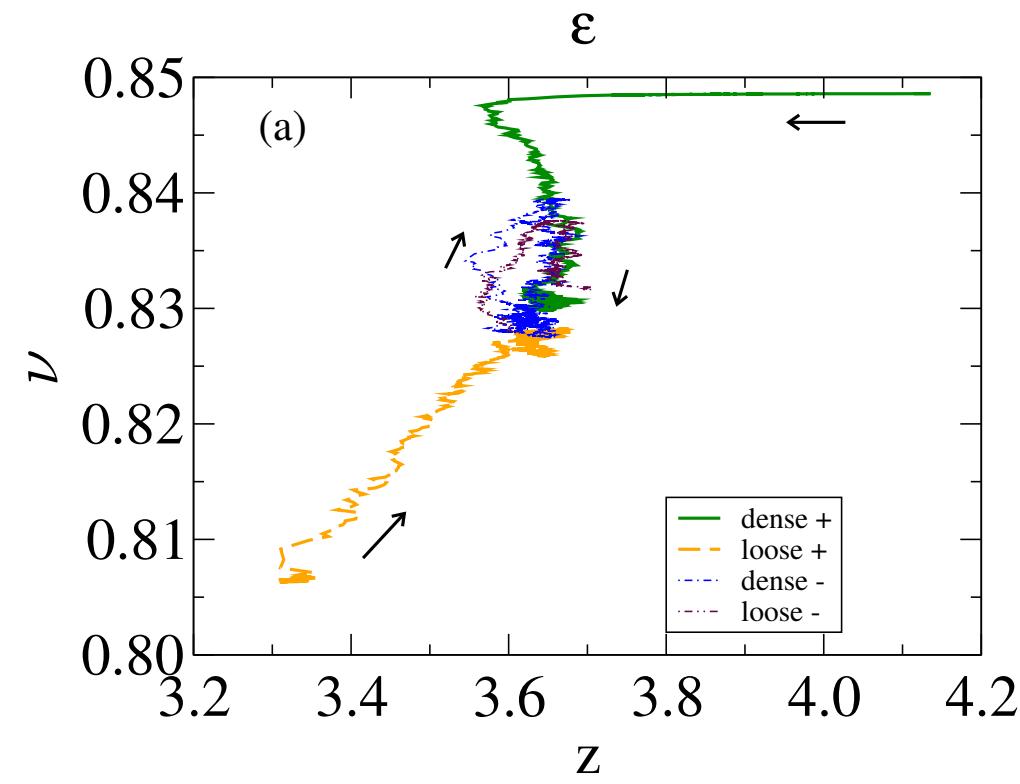
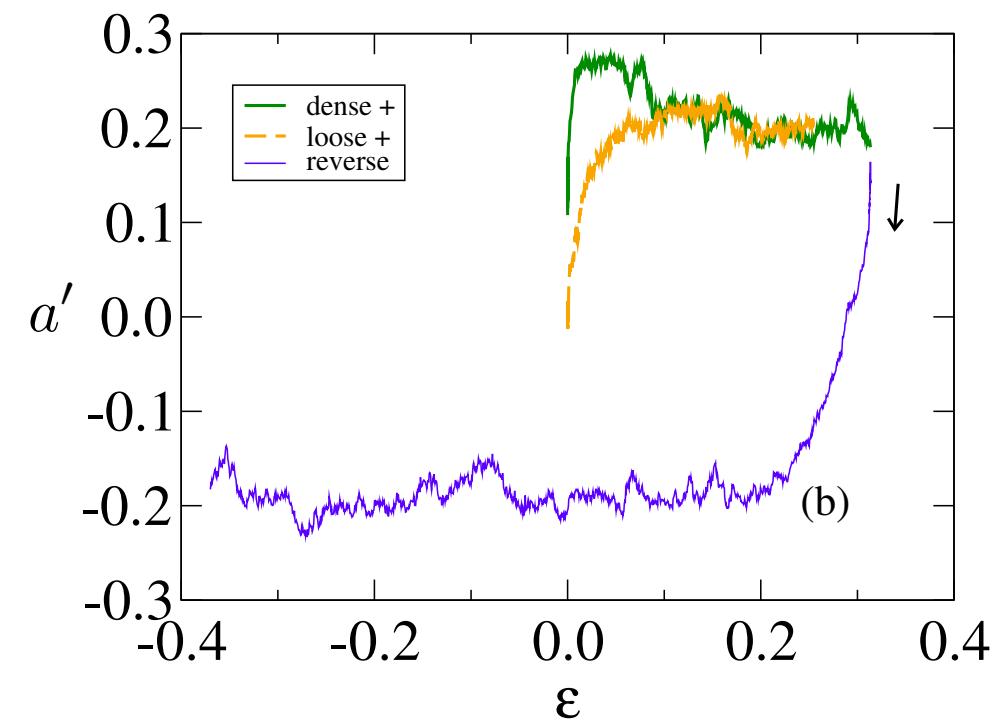
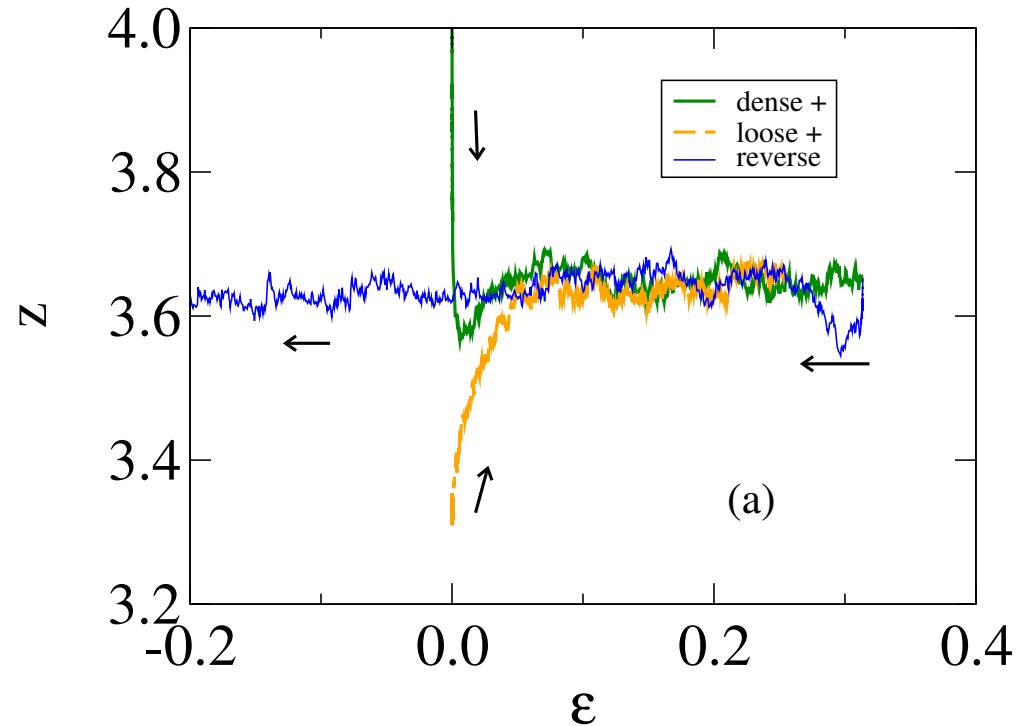


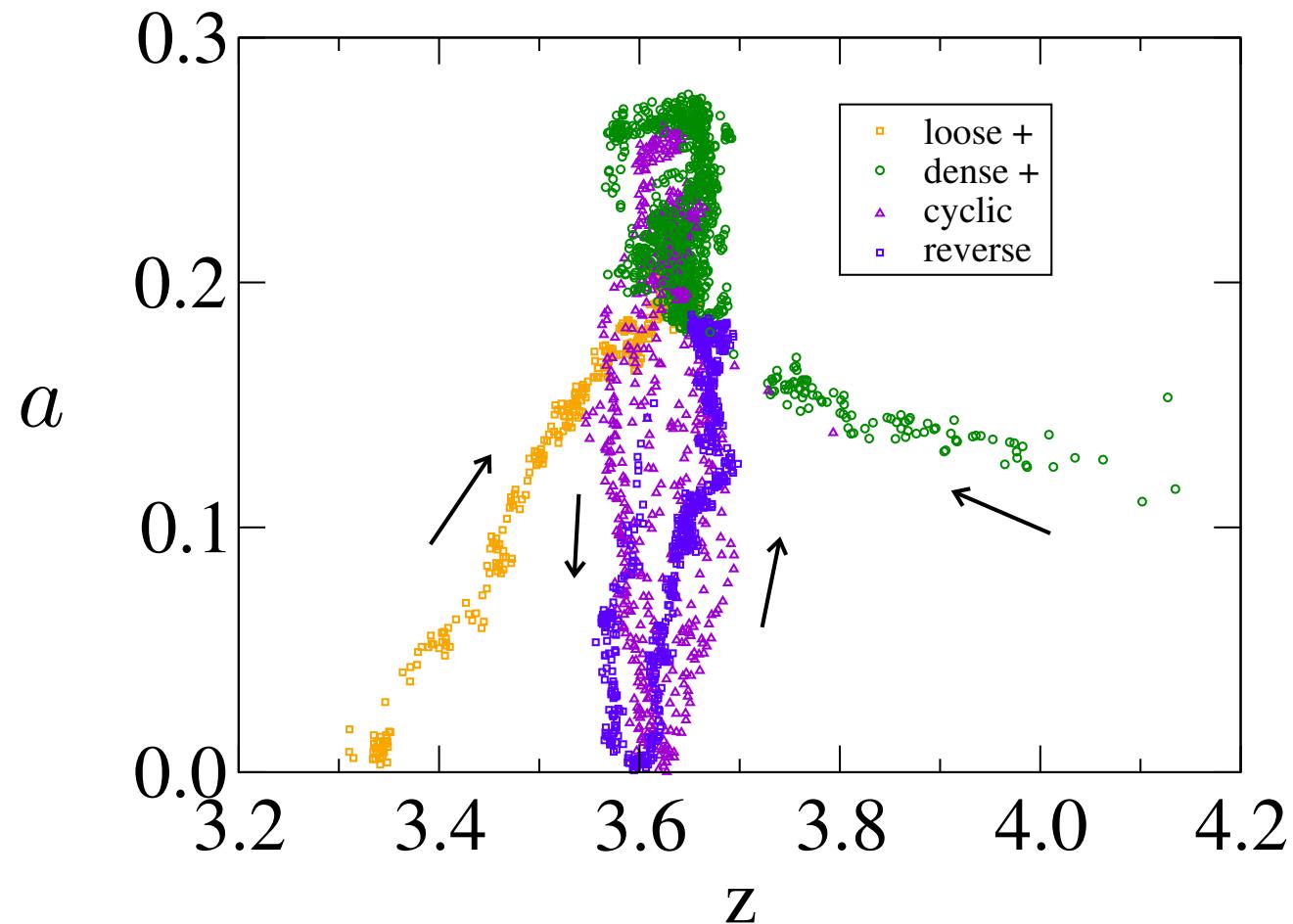


Stress-dilatancy relationship



$$\sin \varphi = \sin \varphi_c + \sin \psi$$





Fabric states

A simple model based on the fabric parameters: $\{z, a_c, \theta_c\}$

$$E(\theta) = \frac{z}{2\pi} \{1 + a_c \cos 2(\theta - \theta_c)\} \quad \text{2D}$$

Steric exclusions \rightarrow upper bound $z \leq z_{max}$
Mechanical equilibrium \rightarrow lower bound $z_{min} \leq z$

\Rightarrow Two limit states:

1) Loosest isotropic state

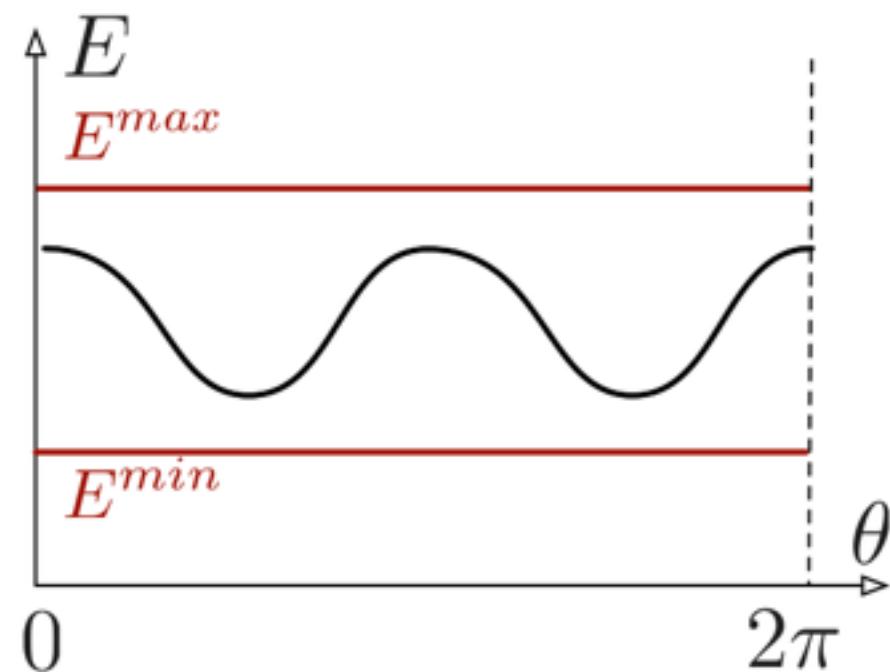
$$E_{max} = \frac{z_{max}}{2\pi}$$

2) Densest isotropic state

$$E_{min} = \frac{z_{min}}{2\pi}$$

Assumption: all other states are enclosed between
the two limit isotropic states

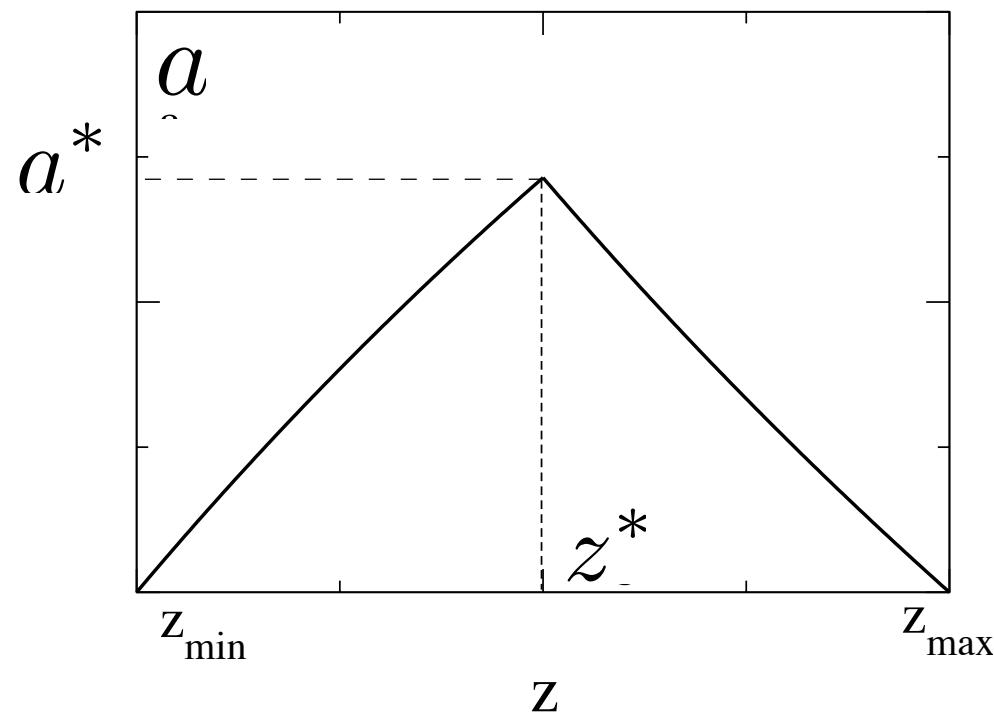
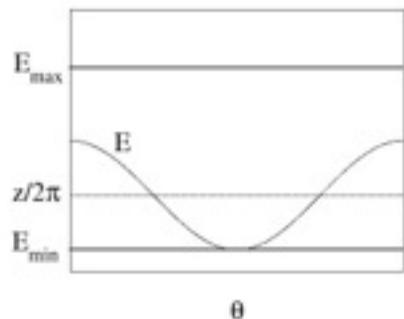
$$E_{min} \leq E \leq E_{max}$$



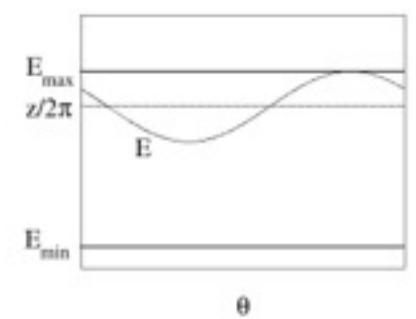
⇒ Upper bound on the anisotropy as a function of z

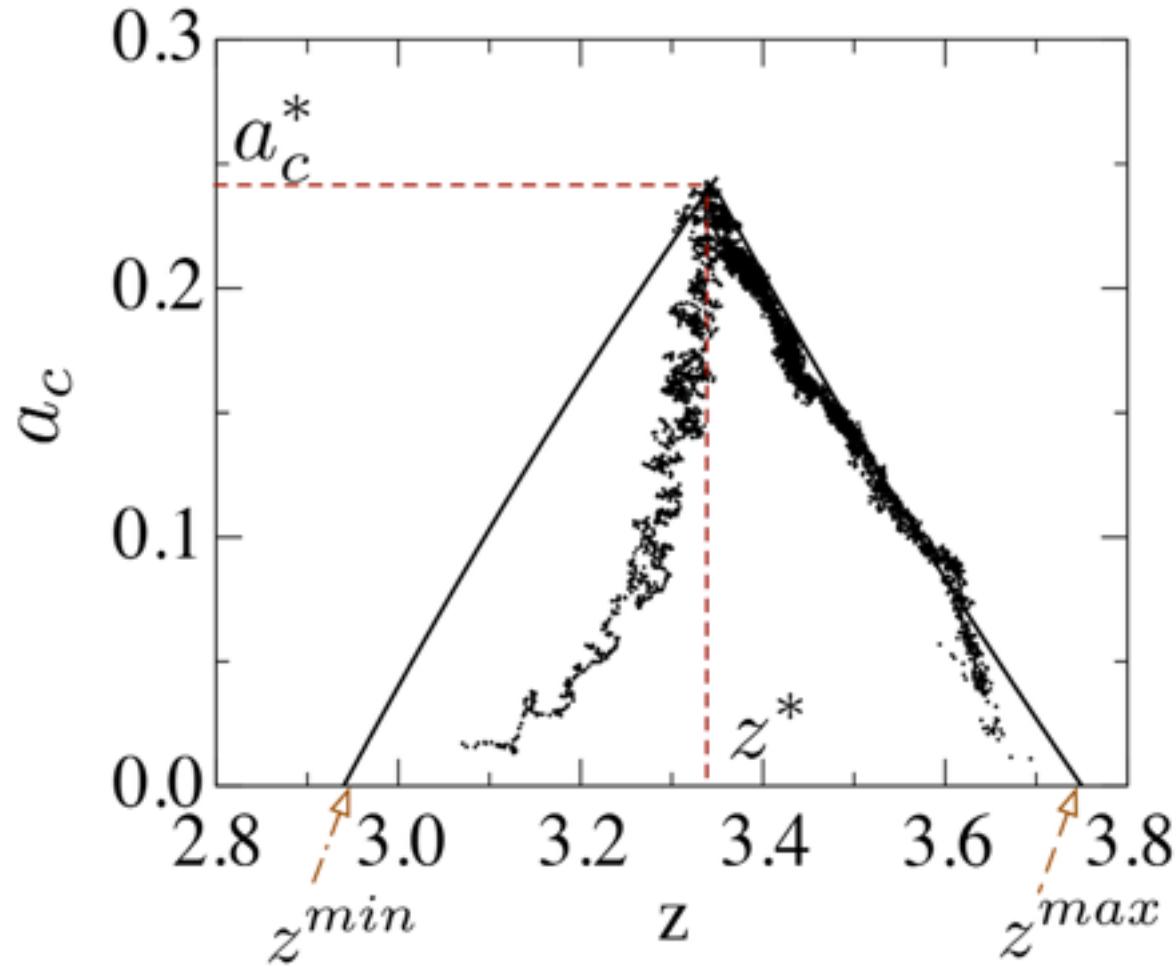
$$a_c^{max}(z) = 2\min \left\{ 1 - \frac{z^{min}}{z}, \frac{z^{max}}{z} - 1 \right\}$$

loss saturation



gain saturation





$$a_c^* = a_c^{max}(z^*) = 2 \frac{a_c^{max} - a_c^{min}}{a_c^{max} + a_c^{min}} \quad z^* = \frac{z^{min} + z^{max}}{2}$$

Inertial critical states

$$\dot{\varepsilon}_q \quad \text{shear rate}$$

$$p \quad \text{mean stress}$$

$$v \sim \dot{\varepsilon}_q d \quad \text{typical collision velocity}$$

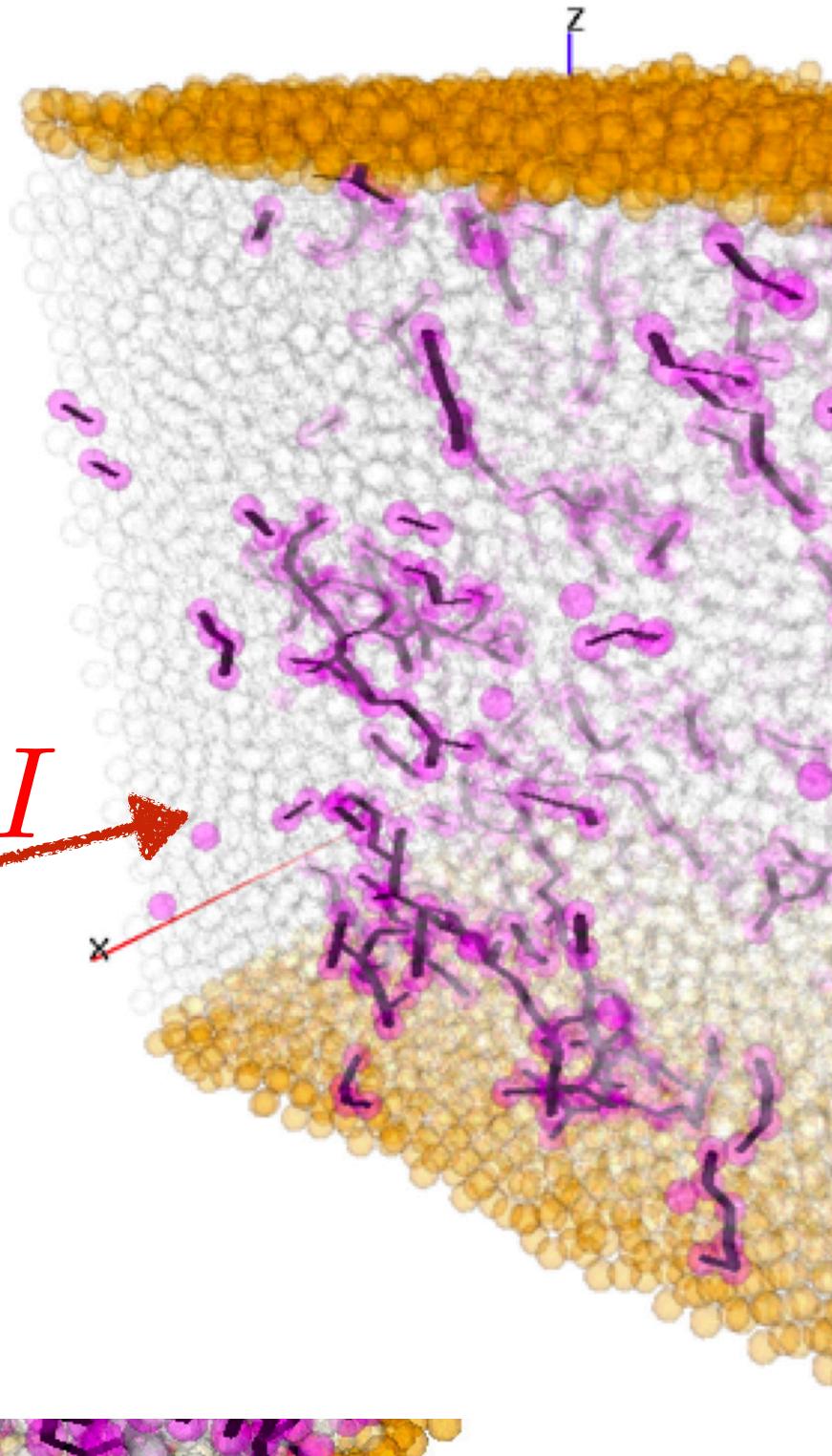
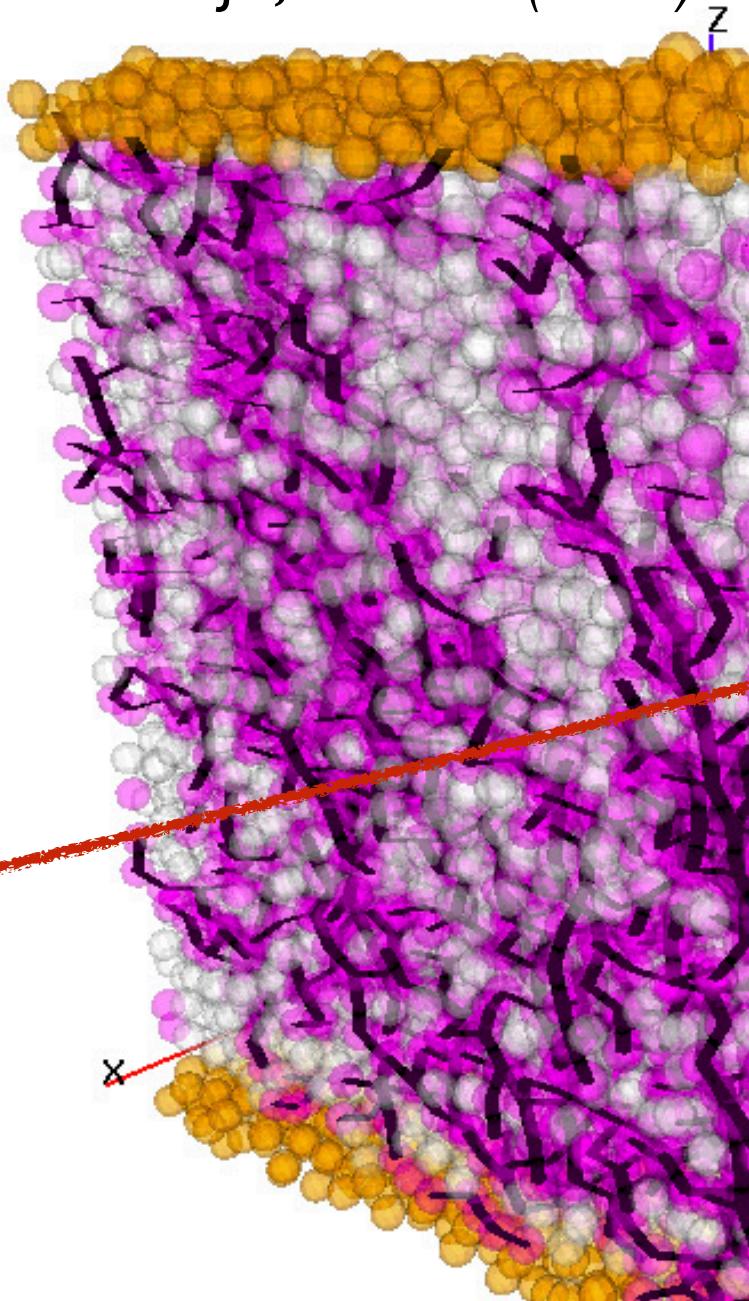
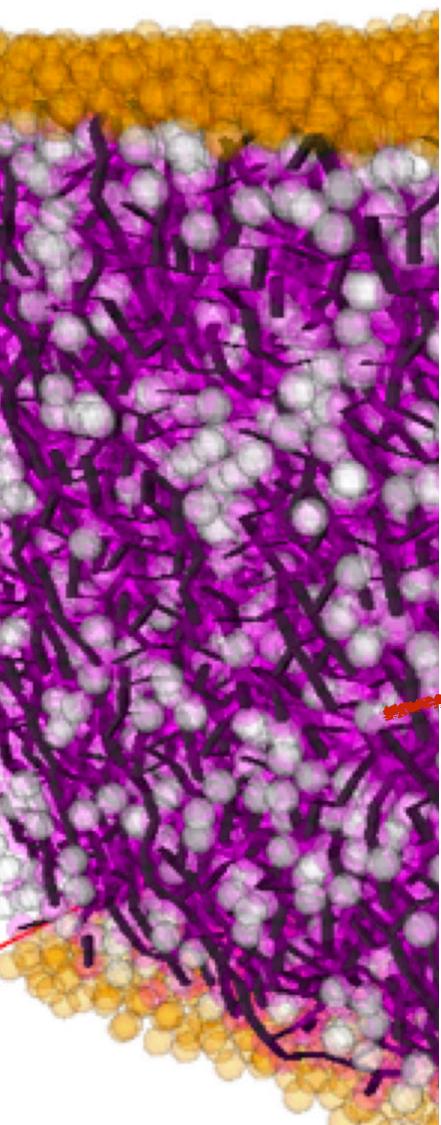
$$f_d \sim mv \times \dot{\varepsilon}_q^{-1} = md\dot{\varepsilon}_q^2 \quad \text{typical impulsive force}$$

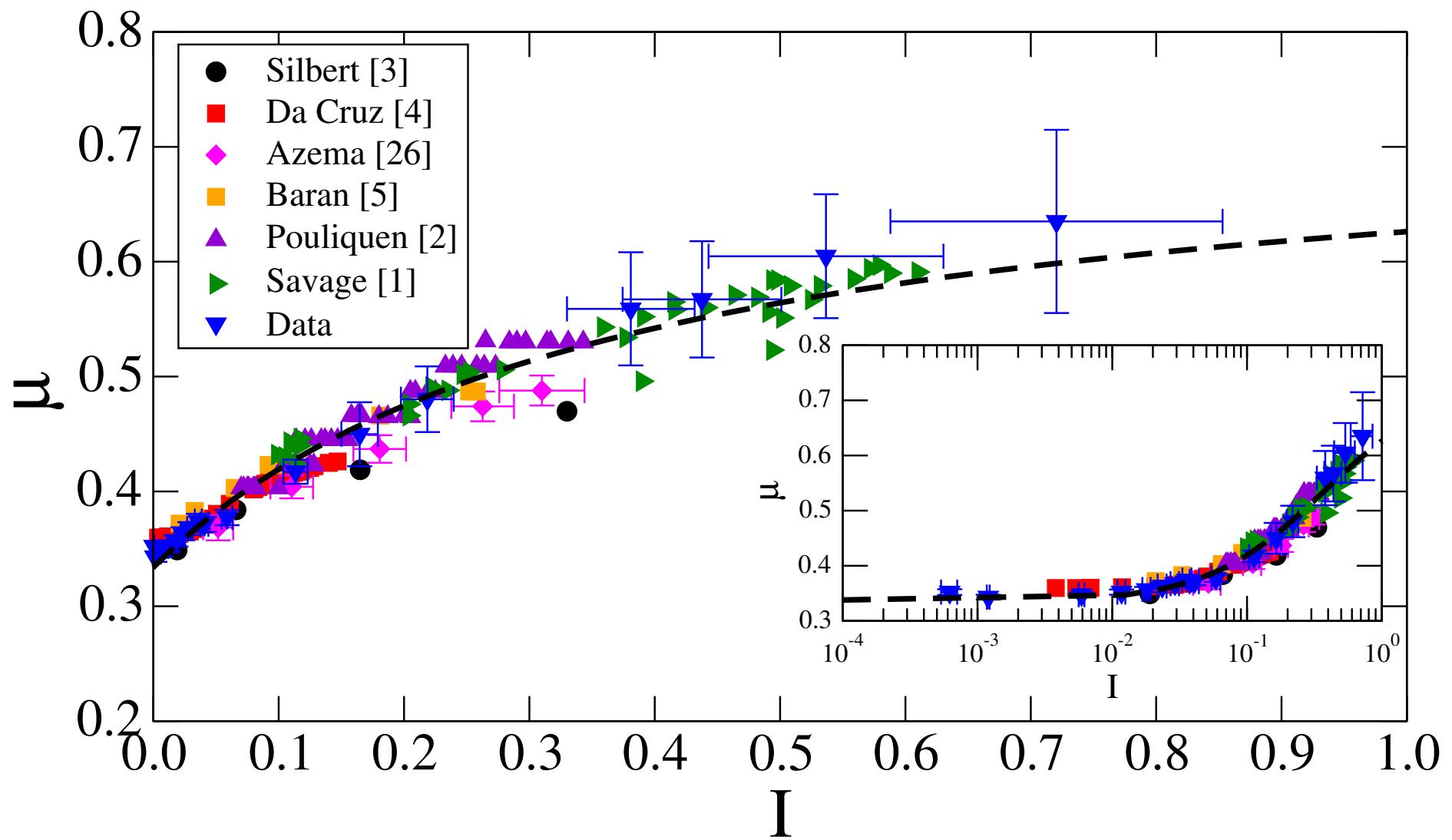
$$f_s = pd^2 \quad \text{typical static force}$$

$$I = \sqrt{\frac{f_d}{f_s}} = \dot{\varepsilon}_q \sqrt{\frac{m}{pd}} \quad \text{Inertial number}$$

3D simulations

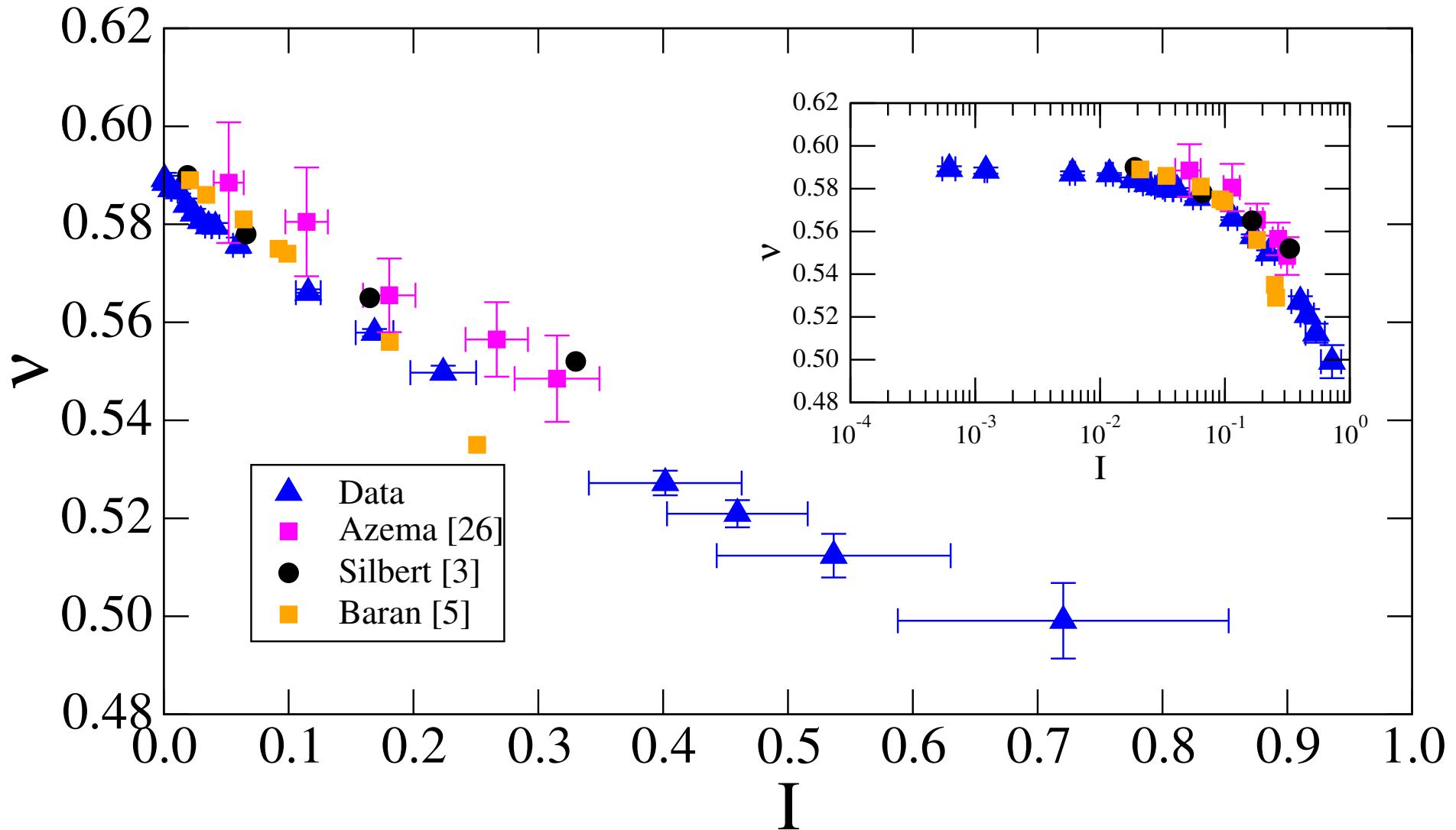
E.Azéma and F.Radjai, PRL 112 (2014)



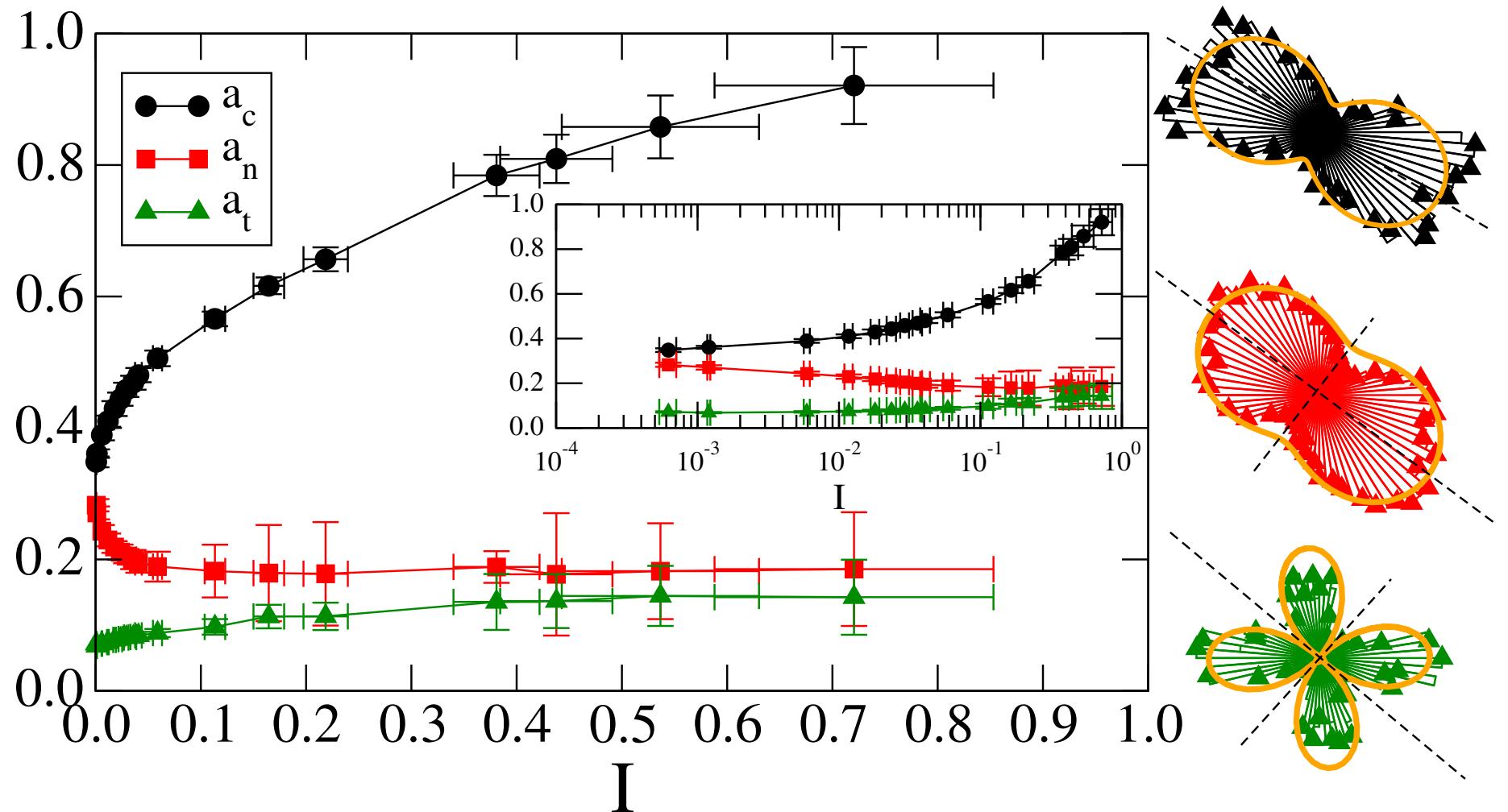


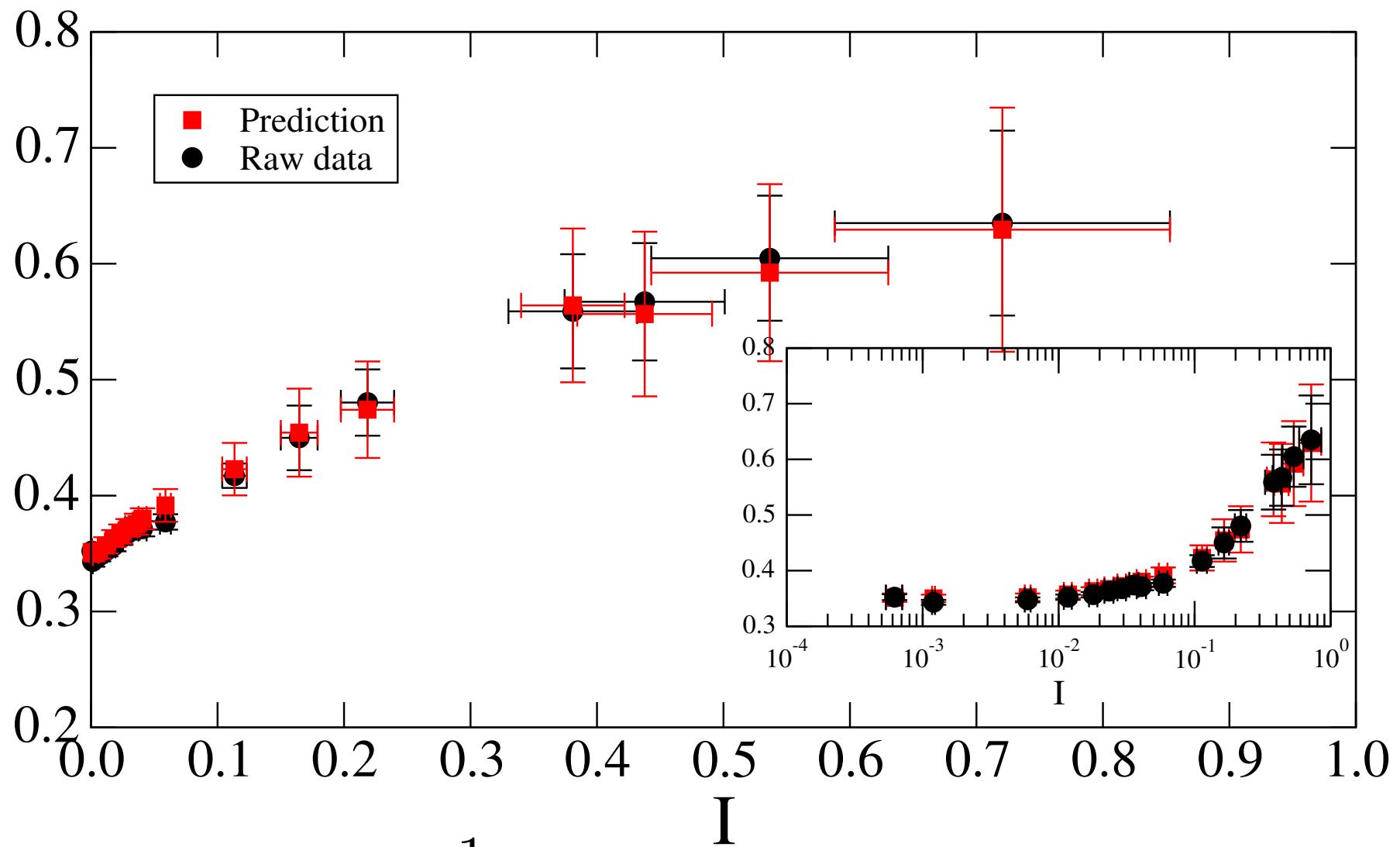
$$\mu = \tan \varphi^*$$

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + I_0/I}$$



$$e \sim -\log \left\{ \frac{p}{p_0} \right\}$$

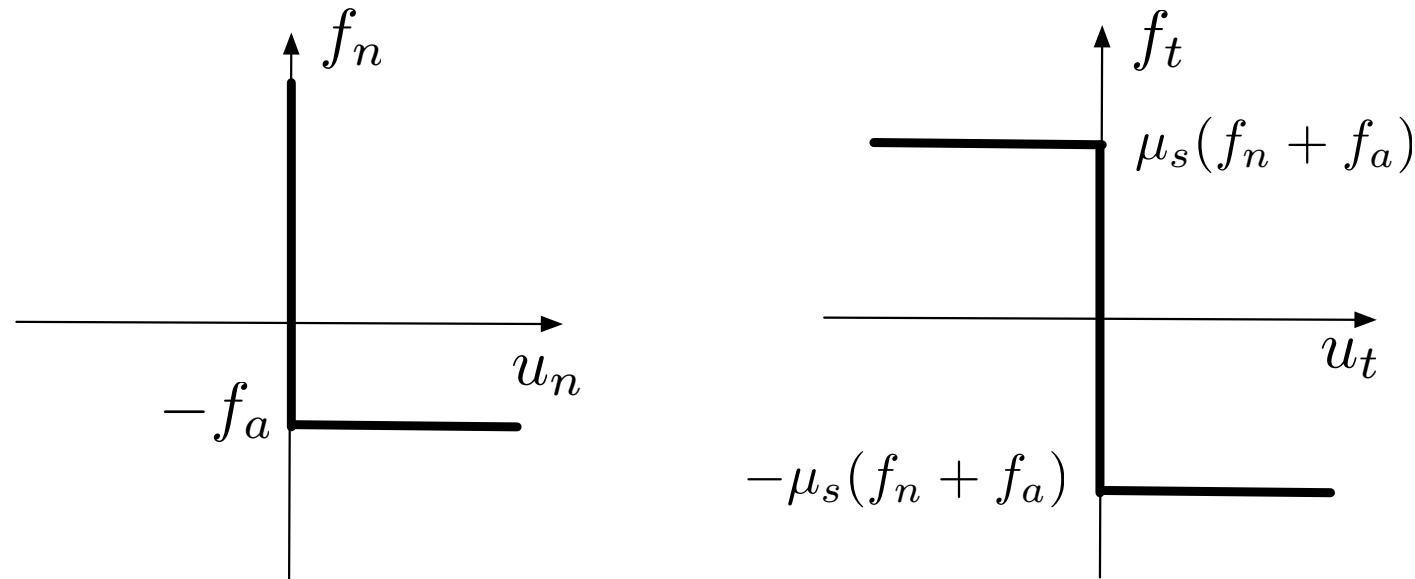




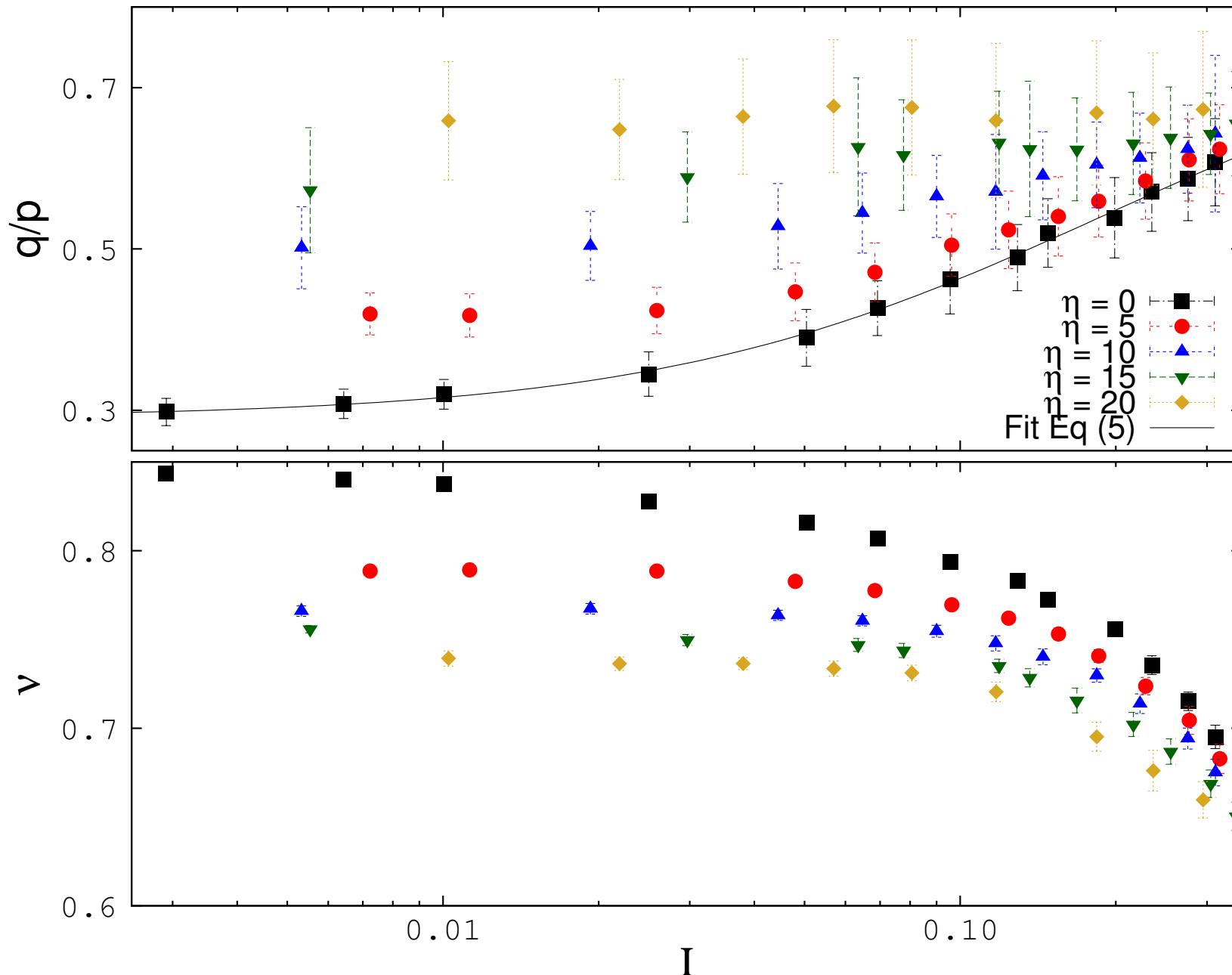
$$\frac{q}{p} = \sin \varphi = \frac{1}{2}(a_c + a_n + a_t)$$

Cohesive critical states

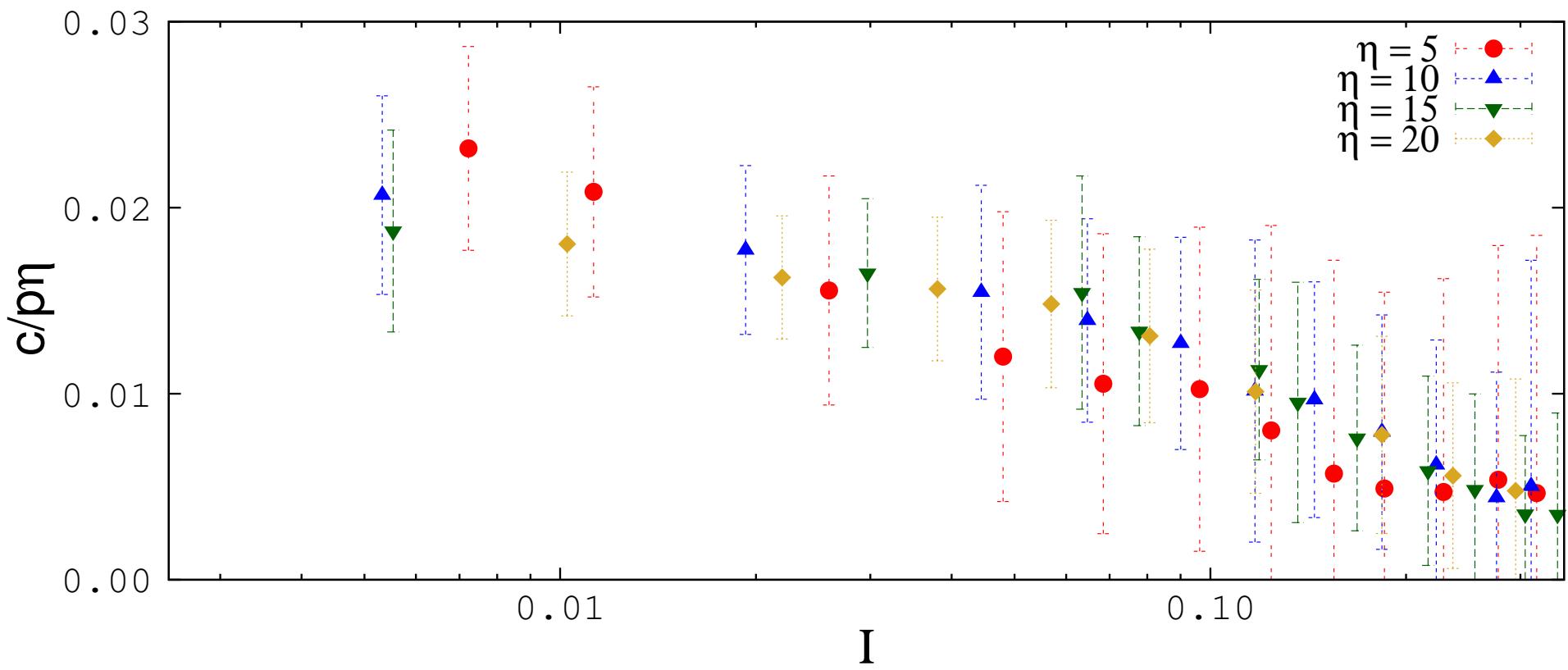
Adhesion: The complementarity relations are shifted.



$$\eta = \frac{f_a}{pd} \quad (2D)$$



$N. Berger, E. Azéma, J.-F. Douce, and F. Radjai (2014)$

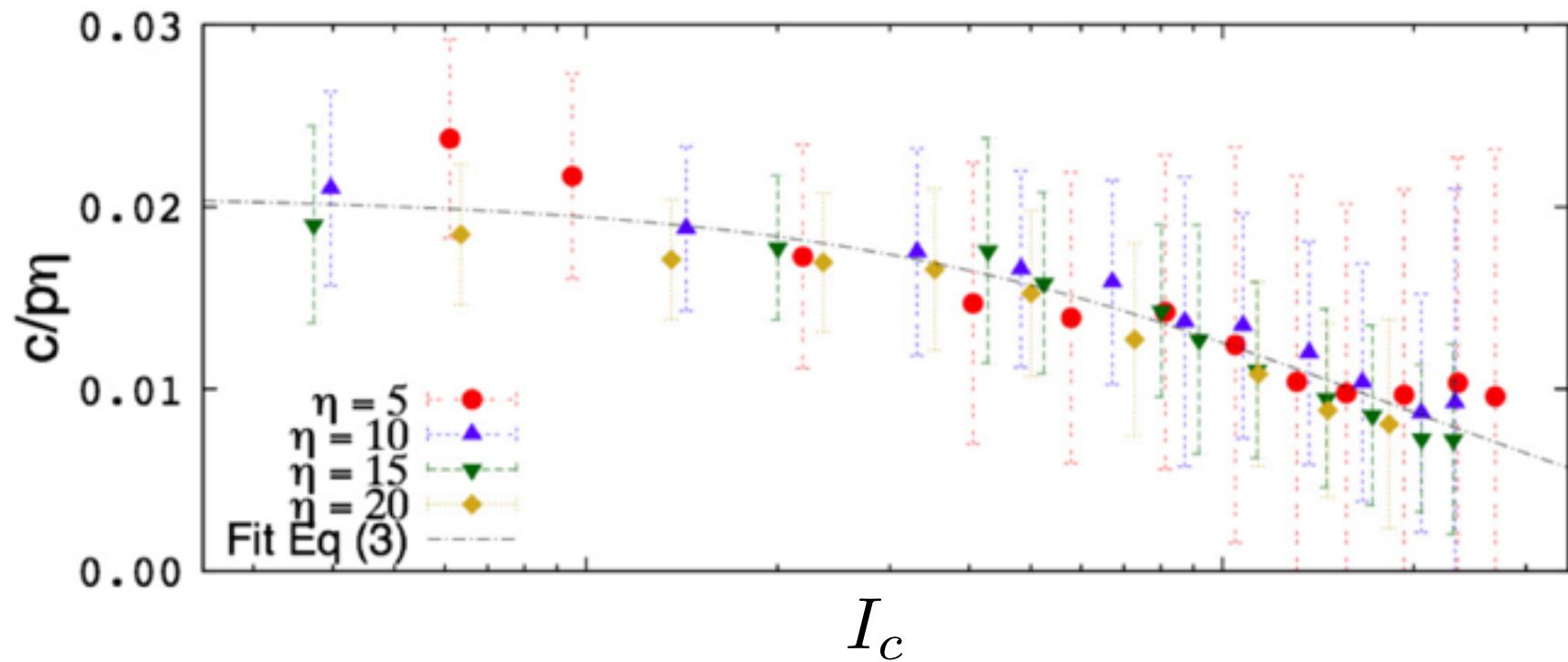


$$\frac{q}{p} = \sin \varphi + \frac{c}{p} \cos \varphi$$

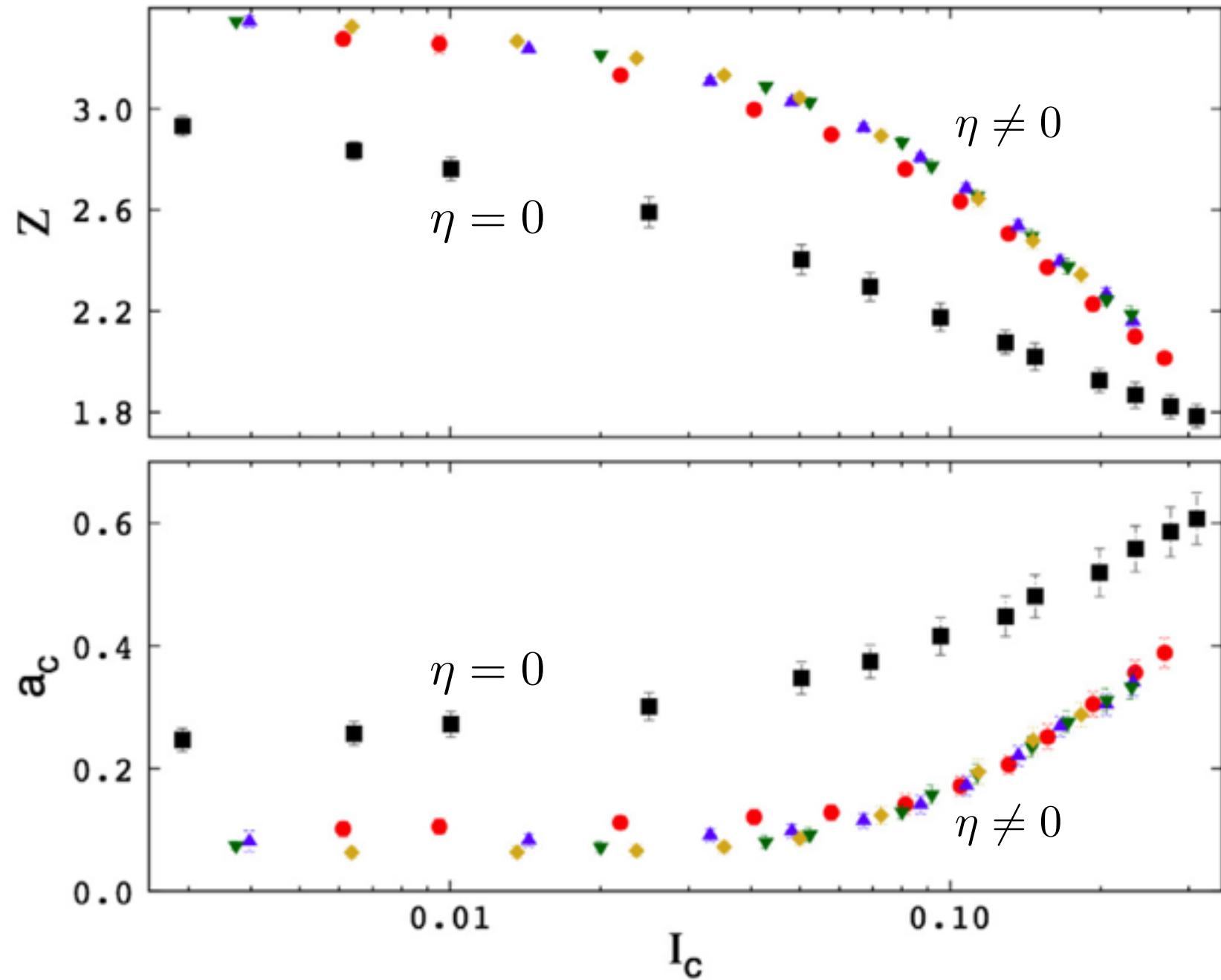
$$\frac{c(I=0)}{p} \sim \eta$$

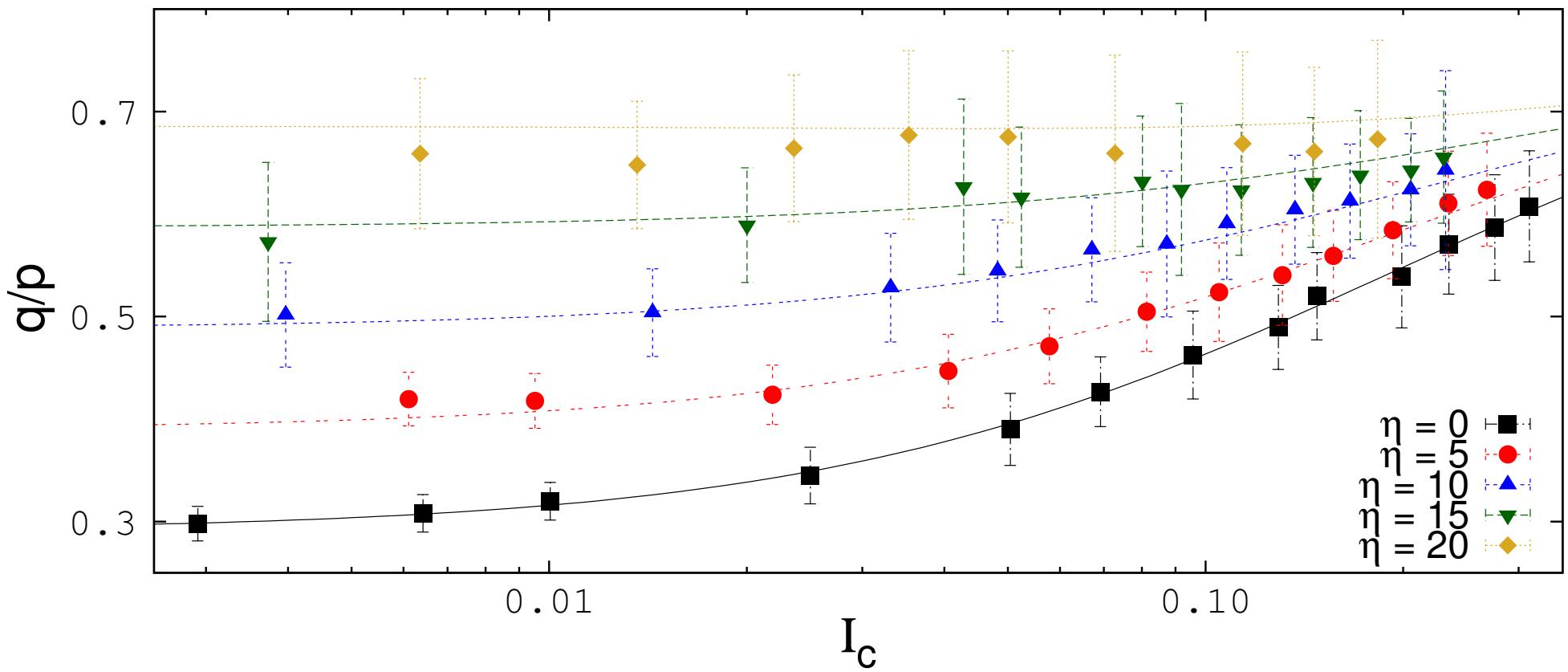
Modified inertial number

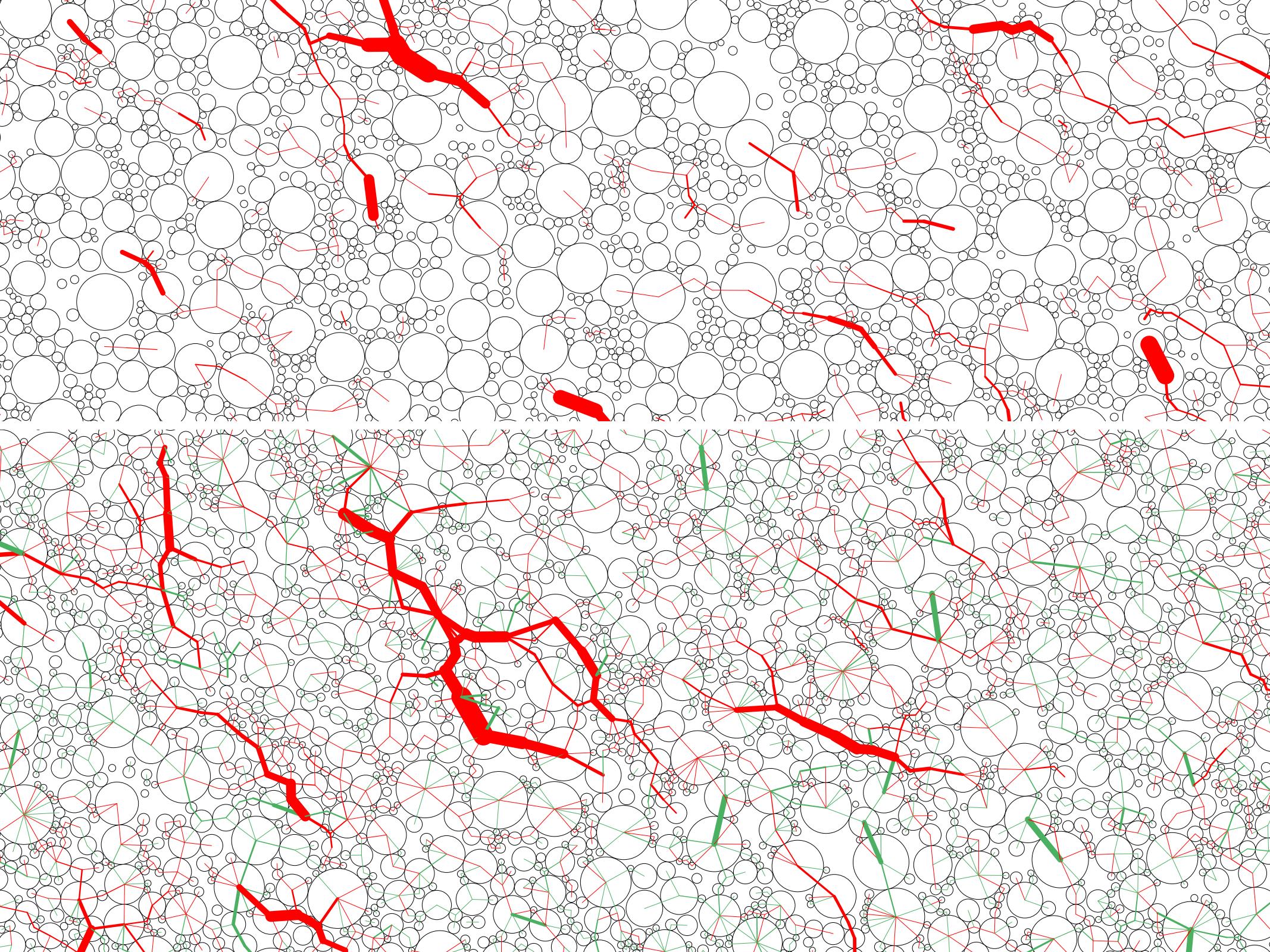
$$f_s = pd + \alpha f_a \quad \Rightarrow \quad I_c = \frac{I}{(1 + \alpha\eta)^{1/2}}$$



$$\frac{c}{p} = \frac{k_0 \eta}{1 - \beta \ln(1 - I_c)}$$







Conclusion

The Contact Dynamics approach provides a general framework for the simulation of plastic deformations of granular materials.

The rheological behavior is reproduced without elastic force laws, thus indicating that the behavior is independent of elastic force law.

The critical state is characterized by both its connectivity and anisotropy.

The critical state may be extended to inertial and cohesive materials. The effect of adhesion is similar to that of particle inertia. Both parameters lead to reduced solid fraction and enhanced shear strength.